The first-order and the deterministic part of Weihrauch degrees

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Weihrauch reducibility (see the survey [1]) is a reducibility notion for multivalued functions between represented spaces. These can be rather complicated and weird objects! In this today, we will explore what we can say when restriction our attention to maps with codomain $\mathbb{N}$, or single-valued functions. The first-order part of a Weihrauch degree f, denoted ${}^1f$ was proposed originally by Dzhafarov, Solomon and Yokoyama; it is the maximal Weihrauch degree below f with a representative having codomain $\mathbb{N}$. The deterministic part $\mathrm{Det}(f)$, formally defined by Goh, P. and Valenti [2], is the maximal degree below f having a single-valued function with codomain $\mathbb{N}^\mathbb{N}$ as a representative. Both are well-defined operations.

These operations can be useful to describe the position of principles "off the beaten track", or principles that are difficult to solve yet have limited uniform strength. Examples include finding an infinite descending sequence in a linear order admitting one [2] or listing all elements in a countable closed set in Cantor space [3], which have computable inputs with no solutions below any given hyperarithmetic degree, yet the deterministic part of both problems is just lim. An example where the first-order part detects such a weakness is finding an isomorphic copy of a line inside a given connected graph, which has non-trivial first-order part yet cannot compute anything non-trivial with finite codomain (jww Cipriani).

Besides concrete applications, there are interesting results about the operations themselves. For example, having non-trivial first-order part is equivalent to having a sequential discontinuity. For the statement, we mention the problem $\mathrm{ACC}_\mathbb{N} : \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}$ where on input 0^ω any number is a valid output, and on input $0^k(n+1)p$, any number except n is a valid output.

Theorem 1 (P. & Solda)**.** The following are equivalent for $f :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}^\mathbb{N}$:

1. $\mathrm{ACC}_\mathbb{N} \leq_\mathrm{W}^* f$.
2. ${}^1f$ is discontinuous.
3. There is a convergent sequence $(a_n)_{n\in\mathbb{N}}$ with $\overline{\{a_n \mid n \in \mathbb{N}\}} \subseteq \mathrm{dom}(f)$ such that $f|_{\overline{\{a_n \mid n\in\mathbb{N}\}}}$ is discontinuous.
4. There exists some discontinuous $g : \{0\} \cup \{2^{-n} \mid n \in \mathbb{N}\} \rightrightarrows \mathbb{N}^\mathbb{N}$ with $g \leq_\mathrm{W}^* f$.
5. There exists some discontinuous $h :\subseteq \mathbb{N}^\mathbb{N} \rightrightarrows \mathbb{N}$ with $h \leq_\mathrm{W}^* f$.

References

[1] Vasco Brattka, Guido Gherardi & Arno Pauly (2021): *Weihrauch Complexity in Computable Analysis*, pp. 367–417. Springer, Cham, doi:10.1007/978-3-030-59234-9_11. Available at https://doi.org/10.1007/978-3-030-59234-9_11. ArXiv 1707.03202.

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[3] Takayuki Kihara, Alberto Marcone & Arno Pauly (202X): *Searching for an analogue of ATR in the Weihrauch lattice*. *Journal of Symbolic Logic*, doi:10.1017/jsl.2020.12.