Random sequences of quantum bits

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This talk covers separate work with
Volkher Scholz, Marco Tomamichel, and Frank Stephan,
and also work by Tejas Bhojraj

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Two ways to study Martin-Löf randomness for objects other than infinite bit sequences

1. Replace Cantor space:
   - General framework: computable probability space as in Hoyrup/Rojas, 2009
   - e.g. space of continuous functions on $[0, 1]$ that vanish at 0 with the Wiener measure
   - easy to adapt the notion of ML-test to this setting.
2. Extend Cantor space
   - A space of generalised bit sequences, along with a notion of ML-test for such generalised sequences.
   - for usual bit sequences, the new notion coincides with the previous one.

Infinite sequences of qubits

- We follow the second approach.
- Study randomness for infinite sequences of qubits.
- We explain the mathematical model for such sequences: states on a certain C^*-algebra M_∞.
- ML-tests can be defined, and are equivalent to the usual tests for classical bit sequences.

- This is joint work with the mathematical physicist Volkher Scholz that started in 2015, and appeared in the J. Math. Physics, 2019.
- Tejas Bhojraj (2021) wrote a thesis on quantum ML-randomness under the supervision of Joseph Miller, and my informal co-supervision. He published one JMP and one TCS paper containing the main results.

Density matrices and states

- A qubit is a physical system that can be in two classical states. E.g. electron with spin up/down.
- A system of n qubits is modelled by a vector in $\mathbb{C}^{2^n}$.
- If one qubit is deleted, the remaining ones enter a statistical superposition of possibilities.
- Such a superposition is modelled by a density matrix: a $2^n \times 2^n$ Hermitian matrix with all eigenvalues positive, and trace 1.

A state is an infinite sequence $(\rho_n)_{n \in \mathbb{N}}$ of $2^n \times 2^n$ density matrices such that deleting the last qubit of ρ_{n+1} yields ρ_n (detail later).

- States such that all the matrices are diagonal are equivalent to measures on Cantor space. They can be seen as statistical superpositions of classical infinite bit sequences.
- This case is easier. We will treat it first.

Martin-Löf absolutely continuous measures

Joint work with Frank Stephan,
STACS paper 2020,
Theoretical Computer Science 900 (2022): 1-19

Martin-Löf's randomness notion (1966)

Let λ denote the uniform (product) measure on Cantor space $\{0,1\}^{\mathbb{N}}$ giving both 0 and 1 the same probability.

- A Martin-Löf test is a uniformly Σ^0_1 sequence $\langle U_m \rangle_{m \in \mathbb{N}}$ of open sets in $\{0,1\}^{\mathbb{N}}$ such that $\lambda U_m \leq 2^{-m}$ for each m.
- A bit sequence Z is Martin-Löf random if Z passes each ML-test, in the sense that Z is only in finitely many of the U_m.

Initial segment complexity

Let $K(x)$ be the length of a shortest prefix free description of a binary string x. Given $Z \in \{0,1\}^{\mathbb{N}}$ and $n \in \mathbb{N}$, let $Z \upharpoonright n$ denote the initial segment $Z(0) \dots Z(n-1)$.

The Schnorr - Levin Theorem says informally that

Z is ML-random $\iff$ each initial segment of Z is incompressible.

Formally:

Theorem (Schnorr 1973, Levin)

Z is ML-random $\iff$ there is $b \in \mathbb{N}$ such that $\forall n\, K(Z \upharpoonright n) \geq n - b$.

Chaitin (1987) proved that the condition $\lim_n K(Z \upharpoonright n) - n = \infty$ is also equivalent to ML-randomness of Z.

Martin-Löf absolutely continuous measures

Recall that a measure μ is absolutely continuous if $\mu(\mathcal{N}) = 0$ for each λ-null set $\mathcal{N}$.

Definition (N. and Stephan, 2022)

A measure μ on $\{0,1\}^{\mathbb{N}}$ is called Martin-Löf absolutely continuous (ML-a.c., for short) if

$$\inf_m \mu(G_m) = 0 \text{ for each Martin-Löf-test } \langle G_m \rangle.$$

Example

- The uniform measure λ is ML-a.c.
- Let $\mu = \sum_k c_k \delta_{Z_k}$ be a positive sum of Dirac measures. Then μ is ML-a.c. $\iff$ all Z_k are Martin-Löf-random.
- In particular, δ_Z is ML-a.c. iff Z is ML-random.

Definition (Recall)

A measure μ on Cantor space is called Martin-Löf absolutely continuous (ML-a.c., for short) if $\inf_m \mu(G_m) = 0$ for each Martin-Löf-test $\langle G_m \rangle$.

- It suffices to consider descending Martin-Löf-tests, because we can replace $\langle G_m \rangle$ by the Martin-Löf-test $\widehat{G}_m = \bigcup_{k>m} G_k$.
- So we can change the passing condition to $\lim_m G_m = 0$.

There is a universal ML-test. So

$$\mu \text{ is ML-a.c.} \iff \mu(\text{non-MLR}) = 0.$$

In particular, each a.c. measure is ML-a.c.

Solovay tests

- A Solovay test is a sequence $\langle \mathcal{S}_k \rangle_{k \in \mathbb{N}}$ of uniformly Σ^0_1 sets such that $\sum_k \lambda \mathcal{S}_k < \infty$. (For ML-tests, we required $\lambda \mathcal{S}_k \leq 2^{-k}$.)
- A measure μ passes such a test if $\lim_k \mu(\mathcal{S}_k) = 0$.

Proposition

A measure μ is ML-a.c. $\iff$ μ passes each Solovay test.

Proof: $\Leftarrow$ is trivial. For $\Rightarrow$:

- by the equivalence of the test notions for bit sequences, the class $\mathcal{V} = \{Z \colon \exists^\infty k\, [Z \in \mathcal{S}_k]\}$ only consists of non-ML random sequences.
- So $\mu(\mathcal{V}) = 0$.
- Using Fatou's Lemma, $\limsup_k \mu(\mathcal{S}_k) \leq \mu(\mathcal{V})$.

Descriptive complexity of initial segments

For a finite bit string x, by $K(x)$ we denote its prefix-free descriptive complexity. Let

$$K(\mu \restriction n) = \textstyle\sum_{|x|=n} K(x)\mu[x].$$

This is the μ-weighted average of the $K(x)$ over all strings x of length n.

Fact

We have $K(\lambda \restriction n) \geq^+ n + K(n)$, were λ denotes the uniform measure.

λ is not the only measure with maximal K-complexity.

Both implications of Levin-Schnorr theorem fail

Proposition (Show that μ is ML-a.c. $\not\Rightarrow K(\mu \restriction n) \geq^+ n$)

There is a ML-a.c. measure μ such that for each $\theta \in (0,1)$, $K(\mu \restriction n) \leq^+ n - n^\theta$. ($\leq^+$ means $\leq$ up to a constant.)

Proof: Define μ as a convex sum $\sum_k c_k \delta_{Z_k}$ for a sequence $\langle Z_k \rangle$ of ML-randoms. The Z_k have long initial segments of 0s, leading to a low $K(\mu \restriction n)$.

Theorem (Show that μ is ML-a.c. $\not\Leftarrow K(\mu \restriction n) \geq^+ n$)

There are bit sequences X, Y such that $\mu = \frac{1}{2}(\delta_X + \delta_Y)$ (avg. of the Dirac measures) satisfies $K(\mu \restriction n) \geq^+ n$, but μ is not ML-a.c.

Proof: Define X and Y so that Y is not ML-random, and $K(X \restriction n) + K(Y \restriction n) \geq^+ 2n$ for all n.

Implications between "randomness notions" for μ

$$
\begin{tikzcd}
\mu \text{ is a.c.} \arrow[ddr] & \forall n\, K(\mu\upharpoonright n) \geq^+ n + K(n) \arrow[d] & \mu \text{ is MLR w.r.t. } \mathbb{P} \arrow[ddl] \\
 & \exists^\infty n\, C(\mu\upharpoonright n) \geq^+ n \arrow[d] & \\
 & \mu \text{ is ML-a.c.} \arrow[d] & \\
 & \lim_n \frac{1}{n} C(\mu\upharpoonright n) = 1 &
\end{tikzcd}
$$

$\mathbb{P}$ is the uniform measure on the space of probability measures. Culver showed that μ is ML-rd. w.r.t. $\mathbb{P} \Rightarrow \mu$ is orthogonal to λ, and hence not a.c. The implications in the middle column of the diagram are strict, via examples that are Dirac measures.

QK complexity for measures on sets of strings

K is a bit unsatisfying for measures. Let α be a measure on the strings of lengths n. Write $n = |\alpha|$. Given $\epsilon > 0$, let

$$QK^\epsilon(\alpha) = \min\{K(F) + \log|F| \colon F \subseteq {}^n2 \wedge \alpha(F) > \epsilon\}.$$

This is the measure case of a Definition of Bhojraj (TCS, 2021).
$K(F)$ means K of the string (of length up to $2^n \cdot n$) encoding F.
The idea is to impose a higher "penalty" on larger sets F.
Trivial upper bound: $QK^\epsilon(\alpha) \leq K(n) + n + O_\epsilon(1)$, via $F = 2^{=n}$.
If α concentrates on a string σ, let $F = \{\sigma\}$ and get $K(\sigma) + O(1)$.

Proposition (Chaitin type condition for weak randomness, Bhojraj)

A measure μ passes all restricted Solovay tests $\iff$
for each ϵ, $\lim_n QK^\epsilon(\mu\upharpoonright n) - n = \infty$.

Restricted Solovay test : a computable sequence $\langle \mathcal{S}_k \rangle_{k\in\mathbb{N}}$ of clopen sets in Cantor space such that $\sum_k \lambda \mathcal{S}_k < \infty$.

Ergodic measures and their entropy

- T denotes the shift operator on $\{0,1\}^\mathbb{N}$. A measure ρ is shift-invariant if $\rho(A) = \rho(T^{-1}(A))$ for each Borel A.
- A shift-invariant measure ρ is ergodic if every measurable set A with $A = T^{-1}(A)$ satisfies $\rho(A) \in \{0,1\}$.
- For ergodic ρ, the entropy $H(\rho)$ is defined as $\lim_n H_n(\rho)$, where
$$H_n(\rho) = -\frac{1}{n} \sum_{|w|=n} \rho[w] \log \rho[w].$$
- $H(\lambda) = 1$.

SMB Theorem (1950s)

Empirical entropy along Z

For $n \geq 0$, for $Z \in \{0,1\}^\mathbb{N}$ let $h_n^\rho(Z) = -\frac{1}{n}\log \rho[Z\upharpoonright n]$.

Note that $H_n(\rho) = E_\rho h_n^\rho$ where E_ρ denotes the expectation w.r.t. ρ.

Theorem (Shannon-McMillan-Breiman theorem)

Let ρ be an ergodic measure.
For ρ-a.e. $Z \in \{0,1\}^\mathbb{N}$ we have $\lim_n h_n^\rho(Z) = H(\rho)$.

Algorithmic version:
If ρ is computable, then the conclusion holds for ρ-ML-random Z by results of Hochman (2009, implicit) and Hoyrup (2012).

Effective SMB theorem for measures

We say that a measure μ is ML-a.c. relative ρ if $\mu(G_m) \to 0$ for each ρ-ML test $\langle G_m \rangle$. Recall that $h_n^\rho(Z) = -\frac{1}{n} \log \rho[Z \upharpoonright n]$.

Proposition

- Let ρ be a computable ergodic measure.
- Suppose that there is D such that $h_n^\rho \leq D$ for each n.

If μ is ML-a.c. w.r.t. ρ then

$$\lim_n E_\mu h_n^\rho = H(\rho),$$

where $E_\mu h_n^\rho = \sum_{|x|=n} h_n^\rho(x)\mu([x])$.

In the paper we give an example based on a renewal process which shows that the boundedness condition is necessary.

Sequences of quantum bits

Finite sequences of quantum bits

- A quantum bit (qubit) $|\phi\rangle$ is in a superposition of the two classical states $|0\rangle$ and $|1\rangle$:
  $|\phi\rangle = \alpha |0\rangle + \beta |1\rangle$, where $\alpha, \beta \in \mathbb{C}$, $|\alpha|^2 + |\beta|^2 = 1$.
- Measurement of a qubit w.r.t. standard basis $|0\rangle, |1\rangle$ yields 0 with probability $|\alpha|^2$, and 1 with probability $|\beta|^2$.
- Let $(\mathbb{C}^2)^{\otimes n}$ (tensor power) be the 2^n-dimensional Hilbert space. The standard basis of $(\mathbb{C}^2)^{\otimes n}$ is given by n-bit strings: it consists of vectors $|a_1 \ldots a_n\rangle := |a_1\rangle \otimes \ldots \otimes |a_n\rangle$.
- The state of n qubits is represented by a unit vector in $(\mathbb{C}^2)^{\otimes n}$.
- This vector is a linear superposition of the base vectors $|a_1 \ldots a_n\rangle$.
- Example: Einstein-Podolsky-Rosen state $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$.

Mixed states, or density operators

- A "pure" state $|\psi\rangle$ is viewed as a unit vector in $(\mathbb{C}^2)^{\otimes n}$. By $|\psi\rangle\langle\psi|$ Dirac denotes the orthogonal projection onto the subspace spanned by $|\psi\rangle$, fixing $|\psi\rangle$.
- A mixed state is a convex linear combination $\sum_{i=1}^{2^n} p_i |\psi_i\rangle\langle\psi_i|$ for pairwise orthogonal pure states ψ_i.
- Recall that for an operator S on a finite dimensional Hilbert space A, the trace is

  $$\mathsf{Tr}(S) = \text{sum of the eigenvalues of } S.$$

- A mixed state is the same as a Hermitean operator S on $(\mathbb{C}^2)^{\otimes n}$ with $\mathsf{Tr}(S) = 1$ and no eigenvalue negative.

Deleting the last qubit

M_{2^n} denotes the set of $2^n \times 2^n$ matrices over $\mathbb{C}$. This is a C^* algebra with the operator norm on matrices. We index the entries by pairs σ, τ of n-bit strings via the reverse binary representation. E.g. $(1010, 1110)$ indexes the entry in position $(5, 7)$.

Partial trace operation $T_n \colon M_{2^{n+1}} \to M_{2^n}$

For a $2^{n+1} \times 2^{n+1}$ matrix $M = (a_{\sigma r, \tau s})$ where $|\sigma|, |\tau| = n$, and r, s are bits, $N = T_n(M)$ is given by the $2^n \times 2^n$ matrix

$$b_{\sigma,\tau} = a_{\sigma 0, \tau 0} + a_{\sigma 1, \tau 1}.$$

Example: Let $n = 1$ and consider the EPR state. We have

$$T_1(\underbrace{\frac{1}{\sqrt{2}}|00\rangle\langle 00| + \frac{1}{\sqrt{2}}|11\rangle\langle 11|)}_{\text{pure state}} = \underbrace{\frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1|}_{\text{mixed state}}.$$

Diagonal states as measures

- As mentioned, diagonal states $S \in M_{2^{n+1}}$ can be identified with measures α on the set of $n+1$-bit strings.
- $\beta = T_n(\alpha)$ is a diagonal state and corresponds to the measure on n-bit strings given by α.
- That is, $\beta([\sigma]) = \alpha([\sigma 0]) + \alpha([\sigma 1])$.

Coherent sequences of density matrices

"Quantum Cantor space" $S(M_\infty)$ consists of the sequences $(\rho_n)_{n \in \mathbb{N}}$ of density matrices in M_{2^n} such that $T_n(\rho_{n+1}) = \rho_n$ for each n.

- This can be identified with the set of states ρ (positive linear functionals of norm 1) on the computable C^* algebra $M_\infty = \varinjlim_n M_{2^n}$.
- Diagonal states (i.e. states such that all ρ_n are diagonal) correspond to measures on Cantor space.
- The tracial state τ is given by $\tau_n = 2^{-n} I_{2^n}$.
- A classical bit sequence Z corresponds to the Dirac measure concentrated on $\{Z\}$. So, Cantor space embeds into $S(M_\infty)$.
- One-dimensional quantum spin system (see 2016 textbook by Naaijkens). Dynamics studied by Bjelakovich et al. (2004).

Algorithmic randomness tests for states

Special projections

- $\mathbb{C}_{\mathtt{alg}}$ denotes the field of algebraic complex numbers.
- A special projection in M_{2^n} is a Hermitian matrix p such that $p^2 = p$, with matrix entries in $\mathbb{C}_{\mathtt{alg}}$.
- Equivalent: subspace of $(\mathbb{C}^2)^{\otimes n}$

Let's discuss the expression $\mathsf{Tr}(\eta\, p)$ where η is a density matrix in M_{2^n}. This is the expected squared length of projecting η onto the range of p. If η is a vector v then $\mathsf{Tr}(\eta\, p) = \langle v \mid p(v)\rangle$. If η is diagonal (i.e., a measure) and p is a set of strings of length n, then $\mathsf{Tr}(\eta\, p)$ equals $\eta(p)$.

Embed M_{2^n} into $M_{2^{n+1}}$ via $A \to A \otimes I_2 = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$.

For (special) projections $p \in M_{2^n}, q \in M_{2^k}$, where $n \le k$, denote by $p \le q$ that the range of p is contained in range of q.

Σ^0_1 probabilistic sets on quantum Cantor space

A quantum Σ^0_1 set G is given by a computable ascending sequence of special projections $\langle p_n \rangle_{n\in\mathbb{N}}$ where $p_n \in M_{2^n}$. For a state $\eta = \langle \eta_n \rangle_{n\in\mathbb{N}}$ let

$$G(\eta) = \sup_n \eta(p_n) = \sup_n \mathsf{Tr}(\eta_n\ p_n).$$

In particular let ρ be the tracial state τ, the diagonal state such that each nonzero entry of τ_n is 2^{-n}. One has

$$G(\tau) = \sup_n 2^{-n}\mathsf{Tr}(p_n).$$

We also write $\tau(G)$ for this value, to stress that it extends the idea of a measure of a set G when all projections correspond to clopen sets.

Quantum Martin-Löf randomness

Recall the "noncommutative measure" $\tau(G) := \sup_n 2^{-n}\mathsf{Tr}(p_n)$.

- A quantum Martin-Löf test is an effective sequence $\langle G_r \rangle_{r\in\mathbb{N}}$ of quantum Σ^0_1 sets such that $\tau(G_r) \le 2^{-r}$ for each r.
- A state ρ passes the test if $\inf_r G_r(\rho) = 0$.

Think of $\langle G_r \rangle_{r\in\mathbb{N}}$ as a sequence of measurements.
The asymptotic measured value at ρ is $\inf_r G_r(\rho)$.

Def. ρ is quantum ML random if it passes each quantum ML test.

There is a universal test (N. and Scholz, 2019).

Theorem (N. and Scholz, Bhojraj)

- Every ML-random bit sequence is quantum ML-random.
- More generally, if a measure ρ is ML-a.c. then ρ is quantum ML random.

Quantum Solovay test

Definition (Quantum Solovay randomness)

- A quantum Solovay test is an effective sequence $\langle G_r \rangle_{r\in\mathbb{N}}$ of quantum Σ^0_1 sets such that $\sum_r \tau(G_r) < \infty$.
- We say that the test is restricted if the G_r are given as projections; that is, from r we can compute n_r and a matrix of algebraic numbers in $M_{2^{n_r}}$ describing G_r.
- We say that ρ is [weakly] quantum Solovay-random if $\lim_r \rho(G_r) = 0$ for each [restricted] quantum Solovay test $\langle G_r \rangle_{r\in\mathbb{N}}$.

Theorem (Bhojraj, 2021)

Quantum Martin-Löf random $\iff$ quantum Solovay random.

The Solovay definition implies that the set of qML states is convex.

Descriptive QK complexity and weak Solovay rd.

Recall that for a measure α be a measure on the strings of lengths n we defined $QK^\epsilon(\alpha) = \min\{K(F) + \log |F| \colon F \subseteq 2^{=|\alpha|} \wedge \alpha(F) > \epsilon\}$.

More generally, let α be a density matrix in M_{2^n}. For $\epsilon > 0$,
$QK^\epsilon(\alpha) = \min\{K(p) + \log |p| \colon$
$p \in M_{2^n}$ is special projection $\wedge \, \mathsf{Tr}(\alpha\, p) > \epsilon\}$.
Here $|p|$ is the dimension of the range of p (Bhojraj, 2021).

Theorem (Bhojraj, 2021)

A state $\rho = \langle \rho_n \rangle$ passes all restricted Solovay tests $\Longleftrightarrow$
for each ϵ, $\lim_n QK^\epsilon(\rho_n) - n = \infty$.

Initial segment characterization of qML-rd.?

- It is unknown whether for a state ρ, passing all restricted Solovay tests is equivalent to qML-randomness. The notions are known to coincide if ρ is non-high.
- Bhojraj showed quantum Schnorr randomness is equivalent to incompressibility w.r.t. a restricted version of QK where the decompression is carried out by computable measure machines. Diagram of some of his results in the JMP and TCS papers:

$$\begin{tikzcd}
\text{q-ML rd.} \arrow[rr, Leftrightarrow, "\text{JMP}"'] \arrow[dd, Rightarrow] & & \text{q-Sol. rd.} \arrow[d, Rightarrow] \\
& \text{QK-incompr.} \arrow[r, Leftrightarrow, "\text{TCS}"'] \arrow[d, Rightarrow] & \text{q-weak Sol. rd.} \\
\text{q-Schnorr rd.} \arrow[r, Leftrightarrow, "\text{TCS}"'] & \text{QK}_C\text{-incompr.} &
\end{tikzcd}$$

Towards an effective quantum SMB theorem

A state ρ on M_∞ is called ergodic if it is an extreme point on the convex set of shift invariant states. Its entropy is

$$h(\rho) = -\lim_n \tfrac{1}{n}\mathsf{Tr}(\rho_n \log \rho_n).$$

- We conjecture that the effective SMB-theorem generalises from measures to the quantum setting: Suppose the empirical entropy of ρ is bounded above. Then $h(\rho) = -\lim_n \frac{1}{n}\mathsf{Tr}(\mu_n \log \rho_n)$, whenever μ is quantum ML-random with respect to ρ.
- By the result for measures, we can do the case that ρ is a measure. To see this one replaces μ by the measure $\overline{\mu}$, where $\overline{\mu}_n$ is the diagonal of μ_n, for each n. (For detail see Logic Blog 2020, Section 9.)
- This also settles the case that ρ is an i.i.d. state, namely, $\rho_n = V^{\otimes n}$ for some fixed 2×2 density matrix V.

Spin chains, and undecidability of the spectral gap

The existence of a spectral gap in the thermodynamical limit is undecidable for finite chains of qudits (d classical states), with behaviour described by local Hamiltonians (Heisenberg model). Due to Bausch, Cubitt, Lucia and Perez-Garcia, 2020.

Let $h^{(1)} \in M_d(\mathbb{C})$ and $h^{(2)} \in M_{d^2}(\mathbb{C})$ be Hermitian matrices, where

- $h^{(1)}$ describes the one-site "interactions", and
- $h^{(2)}$ describes the nearest-neighbour interactions.

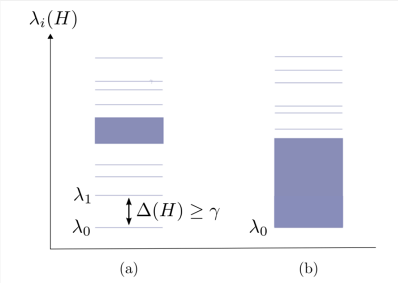
The global Hamiltonian of a spin chain of n qudits is given by shifting the local Hamiltonians and adding up these interactions as the indices vary:

$$H_n = \sum_{i=1}^n h_i^{(1)} + \sum_{i=1}^{n-1} h_{i,i+1}^{(2)}.$$

What is the asymptotic spectral gap?

The spectral gap of a Hamiltonian H acting on a finite-dimensional Hilbert space is $\Delta(H) = \gamma_1(H) - \gamma_0(H)$, the difference between its least two eigenvalues. Suppose that H_n is a Hamiltonian on the d^n-dimensional Hilbert space. The asymptotic spectral gap is

$$\Delta\langle H_n \rangle = \liminf_n \Delta(H_n).$$



From Cubitt et al., Nature 2015. (a) gapped (b) gapless

Coding the halting problem (Bausch et al., 2020)

Given a Turing machine M, Bausch et al. determine a (large) dimension d.

Then, given an input $\eta \in \mathbb{N}$ to M they compute local Hamiltonians $h^{(1)} \in M_d(\mathbb{C})$ and $h^{(2)} \in M_{d^2}(\mathbb{C})$ as above such that

- if $M(\eta)$ halts then the sequence $\langle H_n(\eta) \rangle$ (defined as above by shifting the local interactions) is gapless,
- otherwise the sequence $\langle H_n(\eta) \rangle$ is gapped.

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