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Bi-embeddability relation on computable structures

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New Directions in Computability Theory

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Equivalence relations in computable structure theory

In computable structure theory, structures are usually considered up to isomorophism (or its effective versions).

This talk is a survey of our results and open questions up to **bi-embeddability**.

Computable structure theory

Motivating Question 1

How hard is it to determine that two effectively given structures are equivalent?

Motivating Question 2

Does there exist an equivalent effective copy of a given structure?

Motivating Question 3

If we know that two effectively given structures are equivalent, what does this fact say about the effectiveness properties of the equivalence relation?

Motivating Question 4

Seeing a finite part of a structure from a fixed class, can we determine which of the structures we are observing?

1. Question 1: Classification Problems
2. Question 2: Degree Spectra
3. Question 3: Bi-embeddable categoricity
4. Question 4: Learning structures up to bi-embeddability

Question 1: classification problems

Motivating question 1

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Reducibility

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Definition

Let E, F be equivalence relations on (hyperarithmetical) subsets X, Y of ω respectively. Then E is reducible to F, $E \leq F$ if there exists a partial computable function h, such that $X \subseteq dom(h), h(X) \subseteq Y$ and for all $i, j \in X$,

$$iEj \iff h(i)Fh(j).$$

Bi–embeddability is Σ^1_1 complete

Theorem (F. and S. Friedman)

The equivalence relation of bi-embeddability on computable graphs is Σ^1_1 complete among equivalence relations.

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Question

Let E be an arbitrary Σ^1_1 equivalence relation on ω. Is there a class of structures K with hyperarithmetical $I(K)$ and closed under isomorphism, such that the bi-embeddability relation on K^c is equivalent to E?

Isomorphism

Theorem (F., S. Friedman, Harizanov, Knight, McCoy, Montalbán)

The equivalence relation of isomorphism on computable structures from the following classes is complete for all Σ_1^1 equivalence relations on ω:

1. *graphs and trees,*
2. *torsion–free abelian groups,*
3. *abelian p–groups,*
4. *fields, and others.*

Effective isomorphism

Theorem (F., S. Friedman, Nies)

The computable isomorphism on computable structures is a Σ^0_3 complete equivalence relation for the following classes:

- *trees,*
- *equivalence structures,*
- *Boolean algebras, and others.*

The result relativizes for any computable successor ordinal.

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Theorem (Greenberg, Turetsky)

The relation of hyperarithmetical isomorphism is complete for Π^1_1 equivalence relations.

Open question

Let E be a natural equivalence relation. Assume that for any class K, E on computable structures from K must have complexity Γ (where Γ is $\Sigma^1_1, \Pi^1_1, \Sigma^0_3$, etc.).

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Question

For an arbitrary equivalence relation F of complexity Γ, does there exist a nicely defined class K^c closed under isomorphism, such that the relation E on K^c is equivalent to F?

Question 2: degree spectra

Motivating question 2

Does there exist an equivalent (usually, isomorphic) effective copy of a given structure?

Spectra for equivalence relations

Let $\mathcal{A}$ be a countable structure and E be an equivalence relation on structures.

Definition (F., Semukhin, Turetsky)

The **degree spectrum** of $\mathcal{A}$ **under the relation** E to be

$$\mathrm{DgSp}(\mathcal{A}, E) = \{\deg(D(\mathcal{B})) | \mathcal{B} \text{ is } E-\text{equivalent to } \mathcal{A}\}.$$

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We also call $\mathrm{DgSp}(\mathcal{A}, E)$ the E-spectrum of $\mathcal{A}$.

What are possible degree spectra for various equivalence relations?

Degree spectra of structures

Definition (Richter)

The **degree spectrum of a structure** $\mathcal{M}$ is:

$$\mathrm{DgSp}(\mathcal{A}) = \{\deg(D(\mathcal{B})) \mid \mathcal{A} \cong \mathcal{B}, \mathrm{dom}(\mathcal{B}) = \omega\}.$$

Then $\mathrm{DgSp}(\mathcal{A}) = \mathrm{DgSp}(\mathcal{A}, \cong)$ is the $\cong$-spectrum of $\mathcal{A}$.

Theorem (Knight)

In all nontrivial cases the $\cong$-spectrum is closed upwards.

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Example

- The cone above a degree **d**;
- No countable union of upper cones;
- Slaman; Wehner: all non-computable degrees;
- Greenberg, Montalbán, Slaman: non-hyperarithmetical degrees.

Degree spectra of theories

Definition (U. Andrews, J. Miller)

The **degree spectrum of a theory** T is the set

$$\mathrm{DgSp}(T) = \{\deg(D(\mathcal{B})) | \mathcal{B} \models T\}$$

of Turing degrees of all models of T.

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Example

- Cones are spectra of theories.
- A union of two cones can be a spectrum of a theory.
- All the non-computable degrees form a spectrum of a theory.
- The non-hyperarithmetical degrees do not form a spectrum of a theory.

If $\mathcal{A}$ is a structure, then $\mathrm{DgSp}(\mathrm{Th}(\mathcal{A})) = \mathrm{DgSp}(\mathcal{A}, \equiv)$ is the $\equiv$-spectrum of $\mathcal{A}$.

Degree spectra for $\equiv_{\Sigma_n}$

Definition (F., Semukhin, Turetsky)

For a structure $\mathcal{A}$, define

$$\mathrm{DgSp}(\mathcal{A}, \equiv_{\Sigma_n})) = \{\deg(D(\mathcal{B})) | \Sigma_n\text{−theories of } \mathcal{A} \text{ and } \mathcal{B} \text{ coinside}\}$$

simply the Σ_n**-spectrum** of $\mathcal{A}$.

Example

- For every n, for every **d**, the cone above **d**
- For every $n \geq 2$, a union of two cones may be a Σ_n-spectrum.
- For every $n \geq 2$, the Σ_n-spectrum may consist of exactly the non-computable degrees.
- There exists a structure $\mathcal{A}$ such that its Σ_1-spectrum cannot be presented as a cone above a degree **a**.
- There exists a theory spectrum that is not a Σ_n spectrum.

Bi-embeddability spectra

Let $\approx$ be the relation of bi-embeddability between two countable structures.

Definition

The *bi-embeddability spectrum* of $\mathcal{A}$ is the following set of Turing-degrees

$$\mathrm{DgSp}_{\approx}(\mathcal{A}) = \{\deg(\mathcal{B}) : \mathcal{B} \text{ is bi-embeddable with } \mathcal{A}\}.$$

Observation

$$\mathrm{DgSp}_{\approx}(\mathcal{A}) = \bigcup_{\mathcal{B}\approx\mathcal{A}} (\mathrm{DgSp}_{\cong}(\mathcal{B})).$$

Question

What are the relations between $\cong$-spectra and $\approx$-spectra for countable structures?

Bi-embeddability spectra

Theorem

Every hyperarithmetical structure $\mathcal{A}$ from the following classes is bi-embeddable with a computable structure, thus $\mathbf{0} \in \mathrm{DgSp}_{\approx}(\mathcal{A})$.

1. *Montalbán, 2005: linear orders;*
2. *Greenberg, Montalbán, 2008: abelian p-groups;*
3. *F., Rossegger, San Mauro, 2016: equivalence structures.*

With Rossegger and San Mauro we studied the relations between $\cong$-spectra and $\approx$-spectra for strongly locally finite graphs.

Properties of bi-embeddability spectra

Theorem (F., Rossegger, San Mauro)

- *The $\approx$-spectrum of a structure $\mathcal{A}$ is either a singleton or upwards closed.*
- *A cone above a degree* $\boldsymbol{d}$.
- *All but computable*
- *All the hyperimmune degrees*

Reason: b.e. triviality:

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Reason: b.e. triviality:

Definition

A structure $\mathcal{A}$ is *b.e. trivial* if any bi-embeddable copy $\mathcal{B}$ of $\mathcal{A}$ is isomorphic to $\mathcal{A}$.

Question

Can a union of two cones be realized as a bi-embeddability spectrum?

Strongly locally finite graphs

Definition

A graph $\mathcal{G}$ is *strongly locally finite* if all its components are finite.

Definition

A s.l.f.g. graph $\mathcal{G}$ is *open-ended* if for any of its components C_1 there is a component C_2 such that C_1 is isomorphic to a proper substructure of C_2.

With Rossegger and San Mauro we characterized the isomorphism types of computable open-enden s.l.f.g.'s in terms of sets and functions describing the behaviour of components of the graphs.

Bi-embeddability spectra I

Theorem (FRS)

- *Let $\mathcal{G}$ be open-ended. Then Y computes a bi-embeddable copy of $\mathcal{G}$ if and only if $\mathrm{Tr}(\mathcal{G})$ is c.e. in Y.*

Theorem

The two following facts hold.

1. *There is an open-ended $\mathcal{G}$ such that $\mathrm{DgSp}_{\approx}(\mathcal{G})$ is not a cone of degrees.*
2. *For all open-ended $\mathcal{G}$, $\mathrm{DgSp}'_{\approx}(\mathcal{G}) = \{\mathbf{d}' : \mathbf{d} \in \mathrm{DgSp}_{\approx}(\mathcal{G})\}$ is a cone of degrees.*

Bi-embeddability spectra II

Definition

Let $\mathcal{A}$ be a structure. If there is a structure $\mathcal{B}$ such that $\mathcal{A}$ and $\mathcal{B}$ are bi-embeddable and

$$\mathrm{DgSp}_{\approx}(\mathcal{A}) = \mathrm{DgSp}_{\cong}(\mathcal{B}),$$

then we say that $\mathcal{B}$ is a **bi-embeddability basis** for $\mathcal{A}$.

Theorem

- *If a s.l.f. graph is open-ended, then it has a b.e.-basis.*
- *There is an s.l.f. graph with no b.e.-basis.*

Question

Is there a $\cong$-spectrum that is not a $\approx$-spectrum of any structure? Is there an $\approx$-spectrum that is not a $\cong$-spectrum of any structure? The same question restricted to particular classes of structures.

Elementary bi-embeddability

Denote by $\leftrightarrows$ the relation of *elementary bi-embeddability*.

Rossegger showed that the known counterexamples for isomorphism spectra are also counterexamples for e.b.e. spectra, in particular two cones are impossible.

Theorem (Rossegger)

Let $\mathcal{G}$ be a graph, then there is a graph $\hat{\mathcal{G}}$ such that

$$\mathit{DgSp}_{\leftrightarrows}(\hat{\mathcal{G}}) = \{X : X' \in \mathit{DgSp}_{\approx}(\mathcal{G})\}.$$

Question 3: Bi-embeddable categoricity

Motivating question 3

If we know that two effectively given structures are equivalent, what does this fact say about the effectiveness properties of the equivalence relation?

Classical line of research: computable categoricity (complexity of isomorphisms).

Think about equivalence structures

The equivalence structure $\mathcal{A}$ with infinitely many classes of sizes 1 and 2 is not computably categorical but is $\mathbf{0}'$-categorical.

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On the other hand:

Let $\mathcal{B}$ be a computable equivalence structure with infinitely many classes of size 2 and any number of classes of size 1. Then $\mathcal{A}$ and $\mathcal{B}$ are bi-embeddable, and the embeddings are in fact computable. In other words, $\mathcal{A}$ is *computably bi-embeddably categorical*.

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Definition

A computable structure $\mathcal{A}$ is *computably bi-embeddably categorical* if any computable bi-embeddable copy of $\mathcal{A}$ is bi-embeddable with $\mathcal{A}$ by computable embeddings.

Relative bi-embeddable categoricity

Bazhenov, Fokina, Rossegger, San Mauro:

Definition

A computable structure $\mathcal{A}$ is $\mathbf{0}^{(n)}$ *bi-embeddably categorical* if any computable bi-embeddable copy of $\mathcal{A}$ is bi-embeddable with $\mathcal{A}$ by $\mathbf{0}^{(n)}$ embeddings.

Definition

A countable (not necessarily computable) structure $\mathcal{A}$ is *relatively* $\mathbf{0}^{(n)}$ *bi-embeddably categorical* if for any bi-embeddable copy $\mathcal{B}$, $\mathcal{A}$ and $\mathcal{B}$ are bi-embeddable by $\mathbf{0}^{(n)}$ relative to $\mathcal{A} \oplus \mathcal{B}$ embeddings.

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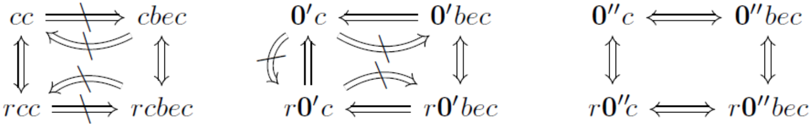
A structure is (relatively) computably b.e. categorical if $n = 0$.

Question

Syntactic description of relative b.e. categoricity?

Categoricity vs. B.e. categoricity



Theorem (Calvert, Cenzer, Harizanov, Morozov)

An equivalence structure is computably categorical iff it is relatively computably categorical.

Theorem (Kach and Turetsky)

There is a $\mathbf{0}'$ *categorical equivalence structure that is not relatively* $\mathbf{0}'$ *categorical.*

Proposition (Calvert, Cenzer, Harizanov, Morozov)

All equivalence structures are relatively $\mathbf{0}''$ *categorical.*

Theorem (BFRS)

All equivalence structures are relatively $\mathbf{0}''$ *bi-embeddably categorical.*

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- *$\mathcal{A}$ has finitely many finite equivalence classes, or*
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Corollary

There is a computably bi-embeddably categorical equivalence structure that is not computably categorical, and vice versa.

$\mathbf{0}'$ and $\mathbf{0}''$ bi-embeddable categoricity

Theorem (BFRS)

A computable equivalence structure $\mathcal{A}$ is $\mathbf{0}'$ b.e. categorical iff

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Theorem (CCHM)

A countable equivalence structure $\mathcal{A}$ is relatively $\mathbf{0}'$ categorical if and only if $\mathcal{A}$ has finitely many infinite equivalence classes or $\mathcal{A}$ is bounded.

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Theorem (BFRS)

Equivalence structures are relatively $\mathbf{0}''$ bi-embeddably categorical.

Degrees of bi-embeddable categoricity

Definition

The **degree of bi-embeddable categoricity** of a computable structure $\mathcal{A}$ is the least Turing degree **d** that, if it exists, computes embeddings between any computable bi-embeddable copies of $\mathcal{A}$.

If, in addition, $\mathcal{A}$ has two computable bi-embeddable copies $\mathcal{A}_0, \mathcal{A}_1$ such that for all embeddings $\mu : \mathcal{A}_0 \hookrightarrow \mathcal{A}_1$, $\nu : \mathcal{A}_1 \hookrightarrow \mathcal{A}_0$, $\mu \oplus \nu \geq_T$ **d**, then **d** is the **strong degree of bi-embeddable categoricity** of $\mathcal{A}$.

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F., Kalimullin, and R. Miller gave the analogous definition for isomorphism.

Theorem (F., Kalimullin, R. Miller)

Every d-c.e. Turing degree is the degree of categoricity of a computable graph.

Degrees of b.e. categoricity of equivalence structures

Theorem (Csima and Ng)

The degree of categoricity of a computable equivalence structure is either $\mathbf{0}, \mathbf{0}'$*, or* $\mathbf{0}''$*.*

Theorem (BFRS)

Let $\mathcal{A}$ *be a computable equivalence structure.*

1. *If* $\mathcal{A}$ *has bounded character and finitely many infinite equivalence classes, then its degree of bi-embeddable categoricity is* $\mathbf{0}$*.*
2. *If* $\mathcal{A}$ *has unbounded character and finitely many infinite equivalence classes, then its degree of bi-embeddable categoricity is* $\mathbf{0}'$*.*
3. *If* $\mathcal{A}$ *has infinitely many infinite equivalence classes, then its degree of bi-embeddable categoricity is* $\mathbf{0}''$*.*

Strongly locally finite graphs

Proposition

Every computable strongly locally finite graph is $\mathbf{0}''$*-categorical.*

Theorem (BFRS)

There is a computable strongly locally finite graph that is not hyperarithmetically bi-embeddably categorical.

Strongly locally finite graphs

Proposition

Every computable strongly locally finite graph is $\mathbf{0}''$-categorical.

Theorem (BFRS)

There is a computable strongly locally finite graph that is not hyperarithmetically bi-embeddably categorical.

Theorem (BFRS)

- *A computable linear order $\mathcal{L}$ is computably b.e. categorical iff $\mathcal{L}$ is finite.*
- *A computable Boolean algebra $\mathcal{B}$ is computably b.e. categorical iff $\mathcal{B}$ is finite.*

Corollary

There exists a computably categorical LO (BA) that is not computably b.e. categorical.

Linear orders and Boolean algebras

Theorem (Bazhenov, Rossegger, Zubkov)

Linear orders of finite Hausdorff rank n are relatively Δ^0_{2n+2} bi-embeddably categorical, but not relatively Δ^0_{2n+1} bi-embeddably categorical.

Theorem (Bazhenov, Rossegger, Zubkov)

Let $\mathcal{B}$ be a computable Boolean algebra. Then $\mathcal{B}$ satisfies one of the following two conditions:

(a) *There is a computable ordinal α such that $\mathbf{0}^{(\alpha)}$ is the degree of b.e. categoricity for $\mathcal{B}$.*

(b) *$\mathcal{B}$ is not hyperarithmetically b.e. categorical, and $\mathcal{B}$ does not have degree of b.e. categoricity.*

Questions

Theorem (Bazhenov)

Every degree $\boldsymbol{d} \geq \mathbf{0}'$*, which contains a* Π^1_0 *function singleton, is a degree of bi-embeddable categoricity.*

Theorem (Csima, Rossegger)

A degree $\boldsymbol{d} \geq \mathbf{0}''$ *is a strong degree of bi-embeddable categoricity iff it is a strong degree of categoricity.*

Question

What are possible degrees of bi-embeddable categoricity?

Question

Is there a degree of b.e.-categoricity that is not a degree of categoricity (and vice versa)?

Question 4: Learning structures up to bi-embeddability

Question

Seeing larger and larger finite pieces of a structure from a fixed class, can we determine which of the structures we are observing?

Learning structures up to bi-embeddability

Combining computable structures and algorithmic learning:

- Let $\mathbb{K}$ be a class of structures with some uniform effective enumeration $\{\mathcal{C}_i\}_{i\in\omega}$ of the computable structures from $\mathbb{K}$, up to isomorphism.

Learning structures up to bi-embeddability

Combining computable structures and algorithmic learning:

- Let $\mathbb{K}$ be a class of structures with some uniform effective enumeration $\{\mathcal{C}_i\}_{i \in \omega}$ of the computable structures from $\mathbb{K}$, up to isomorphism.
- A *learner* **M** is a total function which takes for its inputs finite substructures of a given structure $\S$ from $\mathbb{K}$.

Learning structures up to bi-embeddability

Combining computable structures and algorithmic learning:

- Let $\mathbb{K}$ be a class of structures with some uniform effective enumeration $\{\mathcal{C}_i\}_{i\in\omega}$ of the computable structures from $\mathbb{K}$, up to isomorphism.
- A *learner* **M** is a total function which takes for its inputs finite substructures of a given structure $\S$ from $\mathbb{K}$.
- For an equivalence relation $\sim$, **M InfEx**$_\sim$-*learns* $\mathcal{S}$ if, for all $\mathcal{T} \cong \mathcal{S}$, there exists $n \in \omega$ such that $\mathcal{T} \sim \mathcal{C}_n$ and $\mathbf{M}(\mathcal{T}^i) \downarrow= n$, for all but finitely many i.

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- A family of structures $\mathfrak{A}$ is **InfEx**$_\sim$-*learnable* if there is **M** that learns all $\mathcal{A} \in \mathfrak{A}$.
- **InfEx**$_\sim(\mathbb{K})$ denotes the class of families of $\mathbb{K}$-structures that are **InfEx**$_\cong$-learnable.

Theorem (F., Kötzing, San Mauro)

For the class $\mathbb{K}$ of equivalence structures:

$$\begin{array}{ccc}
\mathbf{InfEx}_{\cong^0} & \xrightarrow{\subsetneq} & \mathbf{InfEx}_{\cong} \\
\subsetneq \big\downarrow & & \big\downarrow \subsetneq \\
\mathbf{InfEx}_{\approx^0} & \xrightarrow{\subsetneq} & \mathbf{InfEx}_{\approx}
\end{array}$$

Question

Investigate b.e.-learnability for other classes.

Question

Investigate b.e.-learnability for other sources of information and convergence behaviour.

Question

Is there a syntactic description of learnability up to bi-embeddability?

Thank you!