Punctual structures relative to oracles

Kalimullin I.Sh.

Kazan Federal University
e-mail:ikalimul@gmail.com

New Directions in Computability Theory
Luminy, France
7-11 March, 2022

Computable structure theory

An algebraic structure $\mathcal{A} = (\mathbb{N}, f_1^{m_1}, \dots, f_M^{m_M}, P_1^{n_1}, \dots, P_N^{n_N})$ is identified with the function

$$F_{\mathcal{A}} = f_1 \oplus \dots \oplus f_M \oplus P_1 \oplus \dots \oplus P_N \equiv \mathcal{A},$$

i.e.,

$$F_{\mathcal{A}}(i, \langle x_1, \dots, x_{m_i} \rangle) = f_i(x_1, \dots, x_{m_i}), \text{ for } 1 \le i \le M,$$

$$F_{\mathcal{A}}(i + M, \langle x_1, \dots, x_{n_i} \rangle) = P_i(x_1, \dots, x_{n_i}), \text{ for } 1 \le i \le N.$$

Computable structure theory

An algebraic structure $\mathcal{A} = (\mathbb{N}, f_1^{m_1}, \dots, f_M^{m_M}, P_1^{n_1}, \dots, P_N^{n_N})$ is identified with the function

$$F_{\mathcal{A}} = f_1 \oplus \cdots \oplus f_M \oplus P_1 \oplus \cdots \oplus P_N \equiv \mathcal{A},$$

i.e.,

$$F_{\mathcal{A}}(i, \langle x_1, \dots, x_{m_i} \rangle) = f_i(x_1, \dots, x_{m_i}), \text{ for } 1 \leq i \leq M,$$

$$F_{\mathcal{A}}(i + M, \langle x_1, \dots, x_{n_i} \rangle) = P_i(x_1, \dots, x_{n_i}), \text{ for } 1 \leq i \leq N.$$

We also usually consider $\mathcal{A} \equiv D(\mathcal{A})$—the atomic diagram of $\mathcal{A}$, An algebraic structure $\mathcal{A} = (\mathbb{N}, f_1^{m_1}, \ldots, f_M^{m_M}, P_1^{n_1}, \ldots, P_N^{n_N})$ is identified with the function

$$F_{\mathcal{A}} = f_1 \oplus \cdots \oplus f_M \oplus P_1 \oplus \cdots \oplus P_N \equiv \mathcal{A},$$

i.e.,

$$F_{\mathcal{A}}(i, \langle x_1, \ldots, x_{m_i} \rangle) = f_i(x_1, \ldots, x_{m_i}), \text{ for } 1 \leq i \leq M,$$

$$F_{\mathcal{A}}(i + M, \langle x_1, \ldots, x_{n_i} \rangle) = P_i(x_1, \ldots, x_{n_i}), \text{ for } 1 \leq i \leq N.$$

We also usually consider $\mathcal{A} \equiv D(\mathcal{A})$—the atomic diagram of $\mathcal{A}$, or replace the operations f_i by their graphs.

Computable structure theory

An algebraic structure $\mathcal{A} = (\mathbb{N}, f_1^{m_1}, \dots, f_M^{m_M}, P_1^{n_1}, \dots, P_N^{n_N})$ is computable if $F_{\mathcal{A}}$ is computable.

An algebraic structure $\mathcal{A} = (\mathbb{N}, f_1^{m_1}, \ldots, f_M^{m_M}, P_1^{n_1}, \ldots, P_N^{n_N})$ is computable if $F_{\mathcal{A}}$ is computable.

An algebraic structure $\mathcal{A} = (\mathbb{N}, f_1^{m_1}, \ldots, f_M^{m_M}, P_1^{n_1}, \ldots, P_N^{n_N})$ is punctual (fully primitive recursive) if $F_{\mathcal{A}}$ is primitive recursive (KMN, 2017).

Computable structure theory

An algebraic structure $\mathcal{A} = (\mathbb{N}, f_1^{m_1}, \dots, f_M^{m_M}, P_1^{n_1}, \dots, P_N^{n_N})$ is computable if $F_{\mathcal{A}}$ is computable.

An algebraic structure $\mathcal{A} = (\mathbb{N}, f_1^{m_1}, \dots, f_M^{m_M}, P_1^{n_1}, \dots, P_N^{n_N})$ is punctual (fully primitive recursive) if $F_{\mathcal{A}}$ is primitive recursive (KMN, 2017).

Note that in the last case we can not replace $F_{\mathcal{A}}$ by a set (i.e., by a $\{0, 1\}$-valued function).

Two definitions of degree spectrum

The degree spectrum of a countable structure $\mathcal{B}$ is usually defined as either

- the collection of Turing degrees of isomorphic copies of $\mathcal{B}$ on the domain $\mathbb{N}$:

  $$\mathbf{DS}(\mathcal{B}) = \{X \mid (\exists \mathcal{A} \cong \mathcal{B})[\text{ the domain of } \mathcal{A} \text{ is } \mathbb{N} \ \& \ \mathcal{A} \equiv_T X]\}, \text{ or}$$

- the collection of Turing oracles which compute an isomorphic copy of $\mathcal{B}$ on the domain $\mathbb{N}$:

  $$\mathbf{DS}(\mathcal{B}) = \{X \mid (\exists \mathcal{A} \cong \mathcal{B})[\text{ the domain of } \mathcal{A} \text{ is } \mathbb{N} \ \& \ \mathcal{A} \leq_T X]\}.$$

Two definitions of degree spectrum

But in most cases two these definitions are the same:

Theorem. (Knight, 1986). Let $\mathcal{B}$ be a structure on the domain $\mathbb{N}$. Then exactly one of the following holds:

- for every $X \geq_T \mathcal{B}$ there is a structure $\mathcal{A} \cong \mathcal{B}$ on the domain $\mathbb{N}$ such that $\mathcal{A} \equiv_T X$;

Two definitions of degree spectrum

But in most cases two these definitions are the same:

Theorem. (Knight, 1986). Let $\mathcal{B}$ be a structure on the domain $\mathbb{N}$. Then exactly one of the following holds:

- for every $X \geq_T \mathcal{B}$ there is a structure $\mathcal{A} \cong \mathcal{B}$ on the domain $\mathbb{N}$ such that $\mathcal{A} \equiv_T X$;
- there is a finite subset $S \subset \mathbb{N}$ such that all permutations of $\mathbb{N}$ which fix S are the automorphisms of $\mathcal{B}$

Two definitions of degree spectrum

But in most cases two these definitions are the same:

Theorem. (Knight, 1986). Let $\mathcal{B}$ be a structure on the domain $\mathbb{N}$. Then exactly one of the following holds:

- for every $X \geq_T \mathcal{B}$ there is a structure $\mathcal{A} \cong \mathcal{B}$ on the domain $\mathbb{N}$ such that $\mathcal{A} \equiv_T X$;
- there is a finite subset $S \subset \mathbb{N}$ such that all permutations of $\mathbb{N}$ which fix S are the automorphisms of $\mathcal{B}$ (in this case all copies $\mathcal{A} \cong \mathcal{B}$ on the domain $\mathbb{N}$ are computable).

Primitive recursive reducibility

A function f is computable in a function g, if there is a Turing machine with the oracle g which computes f ($f \leq_T g$).

Primitive recursive reducibility

A function f is computable in a function g, if there is a Turing machine with the oracle g which computes f ($f \leq_T g$).

A function f is primitive recursive in a function g, if there is a primitive recursive scheme which uses g and produces f ($f \leq_{PR} g$).

Primitive recursive reducibility

A function f is computable in a function g, if there is a Turing machine with the oracle g which computes f ($f \leq_T g$).

A function f is primitive recursive in a function g, if there is a primitive recursive scheme which uses g and produces f ($f \leq_{PR} g$).

Note that in the last case we can not replace functions by sets (e.g., by their graphs).

Primitive recursive reducibility

A function f is computable in a function g, if there is a Turing machine with the oracle g which computes f ($f \leq_T g$).

A function f is primitive recursive in a function g, if there is a primitive recursive scheme which uses g and produces f ($f \leq_{PR} g$).

Note that in the last case we can not replace functions by sets (e.g., by their graphs).

But if f is primitive recursively bounded (i.e., $f(x) \leq p(x)$ for some primitive recursive p) then $f \equiv_{PR} graph(f)$.

Punctual degree spectrum

Theorem. (K, not checked, 2022). There is a primitive recursive permutation $p \neq^* id$ on $\mathbb{N}$ and a computable set $C \subseteq \mathbb{N}$ such that for every permutation q on $\mathbb{N}$ we have

$$(\mathbb{N}, q) \cong (\mathbb{N}, p) \implies q \not\equiv_{PR} C.$$

Punctual degree spectrum

Theorem. (K, not checked, 2022). There is a primitive recursive permutation $p \neq^* id$ on $\mathbb{N}$ and a computable set $C \subseteq \mathbb{N}$ such that for every permutation q on $\mathbb{N}$ we have

$$(\mathbb{N}, q) \cong (\mathbb{N}, p) \implies q \not\equiv_{PR} C.$$

We will use the following definition:

The punctual degree spectrum of a countable structure $\mathcal{B}$ is the collection of primitive recursive oracles which primitive recursively compute an isomorphic copy of $\mathcal{B}$ on the domain $\mathbb{N}$:

$$\mathbf{DS}_{PR}(\mathcal{B}) = \{f \mid (\exists \mathcal{A} \cong \mathcal{B})[\text{ the domain of } \mathcal{A} \text{ is } \mathbb{N} \ \& \ \mathcal{A} \leq_{PR} f]\}.$$

Set basis property

Proposition. (KMM, 2021). For every structure $\mathcal{B}$ on the domain $\mathbb{N}$ there is a primitive recursively bounded $\mathcal{A} \cong \mathcal{B}$ such that $\mathcal{A} \leq_{PR} \mathcal{B}$.

Set basis property

Proposition. (KMM, 2021). For every structure $\mathcal{B}$ on the domain $\mathbb{N}$ there is a primitive recursively bounded $\mathcal{A} \cong \mathcal{B}$ such that $\mathcal{A} \leq_{PR} \mathcal{B}$.

Thus, for every $f \in \mathbf{DS}_{PR}(\mathcal{B})$ there is a set $X \leq_{PR} f$ (i.e., a $\{0,1\}$-valued function) such that $X \in \mathbf{DS}_{PR}(\mathcal{B})$.

Upper cones as the degree spectra (Turing case)

Observation. There is a lot of possibilities to code a set C into a structure $\mathcal{A}_C$ such that

$$\mathbf{DS}(\mathcal{A}_C) = \{X \mid C \leq_T X\}.$$

Upper cones as the degree spectra (Turing case)

Observation. There is a lot of possibilities to code a set C into a structure $\mathcal{A}_C$ such that

$$\mathbf{DS}(\mathcal{A}_C) = \{X \mid C \leq_T X\}.$$

Theorem. (Ash, Knight). Let $\mathcal{A}$ be a structure. A set C is computable in every copy $\mathcal{B} \cong \mathcal{A}$ on the universe $\mathbb{N}$ if and only if for some fixed parameters $\vec{a} \in \mathcal{A}$ there are computable mappings into quantifier-free formulae $n \mapsto \Phi_n$ and $n \mapsto \Psi_n$ such that

$$n \in C \iff \mathcal{A} \models (\exists \vec{x})\Phi_n(\vec{x}, \vec{a}),$$

$$n \notin C \iff \mathcal{A} \models (\exists \vec{x})\Psi_n(\vec{x}, \vec{a}).$$

Upper cones as the degree spectra (punctual case)

Observation. There are possibilities to code a set C into a structure $\mathcal{A}_C$ such that

$$\mathbf{DS}_{PR}(\mathcal{A}_C) = \{f \mid C \leq_{PR} f\}.$$

Upper cones as the degree spectra (punctual case)

Observation. There are possibilities to code a set C into a structure $\mathcal{A}_C$ such that

$$\mathbf{DS}_{PR}(\mathcal{A}_C) = \{f \mid C \leq_{PR} f\}.$$

For example, we can define the finitely generated structure

$$\mathcal{A}_C = (\mathbb{N}, s(x) \equiv x + 1, C(x)).$$

Upper cones as the degree spectra (punctual case)

Observation. There are possibilities to code a set C into a structure $\mathcal{A}_C$ such that

$$\mathbf{DS}_{PR}(\mathcal{A}_C) = \{f \mid C \leq_{PR} f\}.$$

For example, we can define the finitely generated structure

$$\mathcal{A}_C = (\mathbb{N}, s(x) \equiv x + 1, C(x)).$$

Alternatively, we can define the locally finite structure

$$\mathcal{A}_C = (\mathbb{N}, t(x, y) \equiv \min(x + 1, y), C(x)).$$

Upper cones as the degree spectra (punctual case)

Theorem. (KMM, 2021). Let $\mathcal{A}$ be a structure. A set C is primitive recursive in every copy $\mathcal{B} \cong \mathcal{A}$ on the universe $\mathbb{N}$ if and only if for some fixed parameters $\vec{a} \in \mathcal{A}$ there are primitive recursive mappings into quantifier-free formulae $n \mapsto \Phi_n$ and $n \mapsto \Psi_n$ such that for every tuple $\vec{x}$ of pairwisely distinct elements

$$n \in C \iff \mathcal{A} \models \Phi_n(\vec{x}, \vec{a}),$$

$$n \notin C \iff \mathcal{A} \models \Psi_n(\vec{x}, \vec{a}).$$

A reformulation of Ramsey's Theorem states that every infinite relational structure contains an infinite "homogeneous" substructure. Therefore,

Degree spectra of relational structures

A reformulation of Ramsey's Theorem states that every infinite relational structure contains an infinite "homogeneous" substructure. Therefore,

Corollary. (KMM, 2021). If $\mathbf{DS}_{PR}(\mathcal{A}) = \{f \mid C \leq_{PR} f\}$ for a relational structure $\mathcal{A}$ then C is primitive recursive.

A reformulation of Ramsey's Theorem states that every infinite relational structure contains an infinite "homogeneous" substructure. Therefore,

Corollary. (KMM, 2021). If $\mathbf{DS}_{PR}(\mathcal{A}) = \{f \mid C \leq_{PR} f\}$ for a relational structure $\mathcal{A}$ then C is primitive recursive.

Note that, a description of degree spectra of relational structures can be hard.

Degree spectra of relational structures

A reformulation of Ramsey's Theorem states that every infinite relational structure contains an infinite "homogeneous" substructure. Therefore,

Corollary. (KMM, 2021). If $\mathbf{DS}_{PR}(\mathcal{A}) = \{f \mid C \leq_{PR} f\}$ for a relational structure $\mathcal{A}$ then C is primitive recursive.

Note that, a description of degree spectra of relational structures can be hard. An example of computable relational structure without punctual presentations is not straightforward (KMN, 2017).

Unions of upper cones

Folklore Theorem. If $C_1 \mid_T C_2$ then the collection

$$\{X \mid C_1 \leq_T X\} \cup \{X \mid C_2 \leq_T X\}$$

is not a degree spectrum of a structure.

Unions of upper cones

Folklore Theorem. If $C_1 \mid_T C_2$ then the collection

$$\{X \mid C_1 \leq_T X\} \cup \{X \mid C_2 \leq_T X\}$$

is not a degree spectrum of a structure.

The similar forcing arguments give

Theorem. If $C_1 \mid_{PR} C_2$ then the collection

$$\{f \mid C_1 \leq_{PR} f\} \cup \{f \mid C_2 \leq_{PR} f\}$$

is not a punctual degree spectrum of a structure.

Complements of lower cones (Turing case)

Theorem. (Slaman, Wehner, 1999). There is a structure $\mathcal{A}$ such that

$$\mathbf{DS}(\mathcal{A}) = \{X \mid X \not\leq_T \emptyset\}.$$

Complements of lower cones (Turing case)

Theorem. (Slaman, Wehner, 1999). There is a structure $\mathcal{A}$ such that

$$\mathbf{DS}(\mathcal{A}) = \{X \mid X \not\leq_T \emptyset\}.$$

Theorem. (K, 2008). There is a structure $\mathcal{A}$ such that

$$\mathbf{DS}(\mathcal{A}) = \{X \mid X \not\leq_T \emptyset'\}.$$

Complements of lower cones (Turing case)

Theorem. (Slaman, Wehner, 1999). There is a structure $\mathcal{A}$ such that

$$\mathbf{DS}(\mathcal{A}) = \{X \mid X \not\leq_T \emptyset\}.$$

Theorem. (K, 2008). There is a structure $\mathcal{A}$ such that

$$\mathbf{DS}(\mathcal{A}) = \{X \mid X \not\leq_T \emptyset'\}.$$

Theorem. (ACKLMM, 2016). There is no structure $\mathcal{A}$ such that

$$\mathbf{DS}(\mathcal{A}) = \{X \mid X \not\leq_T \emptyset^{(n)}\},$$

where $n \geq 2$.

Complements of lower cones (Turing case)

Theorem. (Slaman, Wehner, 1999). There is a structure $\mathcal{A}$ such that

$$\mathbf{DS}(\mathcal{A}) = \{X \mid X \not\leq_T \emptyset\}.$$

Complements of lower cones (Turing case)

Theorem. (Slaman, Wehner, 1999). There is a structure $\mathcal{A}$ such that

$$\mathbf{DS}(\mathcal{A}) = \{X \mid X \not\leq_T \emptyset\}.$$

In fact, Wehner used a coding the family

$$\mathcal{W} = \{\{n\} \oplus F \mid F \text{ is finite } \&\ F \neq W_n\}$$

into a structure, where W_n is the n-th c.e. set.

Complements of lower cones (punctual case)

Question. Is there a structure $\mathcal{A}$ such that

$$\mathbf{DS}_{PR}(\mathcal{A}) = \{f \mid f \not\leq_{PR} \emptyset\}?$$

Complements of lower cones (punctual case)

Question. Is there a structure $\mathcal{A}$ such that

$$\mathbf{DS}_{PR}(\mathcal{A}) = \{f \mid f \not\leq_{PR} \emptyset\}?$$

Question. For which functions g there is a structure $\mathcal{A}$ such that

$$\mathbf{DS}_{PR}(\mathcal{A}) = \{f \mid f \not\leq_{PR} g\}?$$

Testing the set basis property

Proposition. (K, 2022). If a function g is not primitive recursive then there is a set $X \leq_{PR} g$ which is not primitive recursive.

Testing the set basis property

Proposition. (K, 2022). If a function g is not primitive recursive then there is a set $X \leq_{PR} g$ which is not primitive recursive. So the test passes for $g = \emptyset$.

Testing the set basis property

Proposition. (K, 2022). If a function g is not primitive recursive then there is a set $X \leq_{PR} g$ which is not primitive recursive. So the test passes for $g = \emptyset$.

Let h be a function which is not bounded by a primitive recursive function.

Testing the set basis property

Proposition. (K, 2022). If a function g is not primitive recursive then there is a set $X \leq_{PR} g$ which is not primitive recursive. So the test passes for $g = \emptyset$.

Let h be a function which is not bounded by a primitive recursive function. Let $\{P_n^h\}_{n \in \mathbb{N}}$ be a Gödel numbering of all sets $P \leq_{PR} h$.

Testing the set basis property

Proposition. (K, 2022). If a function g is not primitive recursive then there is a set $X \leq_{PR} g$ which is not primitive recursive. So the test passes for $g = \emptyset$.

Let h be a function which is not bounded by a primitive recursive function. Let $\{P_n^h\}_{n \in \mathbb{N}}$ be a Gödel numbering of all sets $P \leq_{PR} h$. Then for the "universal" set

$$U^h = \{\langle n, m \rangle \mid m \in P_n^h\}$$

we have $h \not\leq_{PR} U^h$ but $X \leq_{PR} U^h$ for every set $X \leq_{PR} h$.

Testing the set basis property

Proposition. (K, 2022). If a function g is not primitive recursive then there is a set $X \leq_{PR} g$ which is not primitive recursive. So the test passes for $g = \emptyset$.

Let h be a function which is not bounded by a primitive recursive function. Let $\{P_n^h\}_{n \in \mathbb{N}}$ be a Gödel numbering of all sets $P \leq_{PR} h$. Then for the "universal" set

$$U^h = \{\langle n, m \rangle \mid m \in P_n^h\}$$

we have $h \not\leq_{PR} U^h$ but $X \leq_{PR} U^h$ for every set $X \leq_{PR} h$.

Proposition. (K, 2022). The test fails if $g = U^h$.

Testing the set basis property

Proposition. (K, 2022). If a function g is not primitive recursive then there is a set $X \leq_{PR} g$ which is not primitive recursive. So the test passes for $g = \emptyset$.

Let h be a function which is not bounded by a primitive recursive function. Let $\{P_n^h\}_{n\in\mathbb{N}}$ be a Gödel numbering of all sets $P \leq_{PR} h$. Then for the "universal" set

$$U^h = \{\langle n, m\rangle \mid m \in P_n^h\}$$

we have $h \not\leq_{PR} U^h$ but $X \leq_{PR} U^h$ for every set $X \leq_{PR} h$.

Proposition. (K, 2022). The test fails if $g = U^h$. So the collection

$$\{f \mid f \not\leq_{PR} U^h\}$$

is not the punctual degree spectrum of a structure.

Coding a family into a structure

Let $\mathcal{F}$ be a countable family of subsets of $\mathbb{N}$.

Coding a family into a structure

Let $\mathcal{F}$ be a countable family of subsets of $\mathbb{N}$. Define the structure $\mathcal{A}_{\mathcal{F}}$ on the domain $\mathbb{N} \times \mathbb{N} \times \mathcal{F}$ with the unary operations

$$r(x, y, U) = (0, y, U),$$

$$s(x, y, U) = (x + 1, y, U),$$

and the unary predicate

$$P(x, y, U) = \text{``} x \in U \text{''},$$

where $x, y \in \mathbb{N}$ and $U \in \mathcal{F}$.

Coding a family into a structure

Let $\mathcal{F}$ be a countable family of subsets of $\mathbb{N}$. Define the structure $\mathcal{A}_{\mathcal{F}}$ on the domain $\mathbb{N} \times \mathbb{N} \times \mathcal{F}$ with the unary operations

$$r(x, y, U) = (0, y, U),$$

$$s(x, y, U) = (x + 1, y, U),$$

and the unary predicate

$$P(x, y, U) = \text{“}x \in U\text{”},$$

where $x, y \in \mathbb{N}$ and $U \in \mathcal{F}$.

Proposition. (K, 2022). $f \in \mathbf{DS}_{PR}(\mathcal{A}_{\mathcal{F}})$ iff there is a $Y \leq_{PR} f$ such that

$$\mathcal{F} = \{Y^{(n)} \mid n \in \mathbb{N}\}.$$

An analogue of the result of Wehner, 1999

Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be the Gödel numbering of all partially computable functions, and let $\{P_n\}_{n\in\mathbb{N}}$ be the Gödel numbering of all primitive recursive sets.

An analogue of the result of Wehner, 1999

Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be the Gödel numbering of all partially computable functions, and let $\{P_n\}_{n\in\mathbb{N}}$ be the Gödel numbering of all primitive recursive sets. Then we can define the family

$$\mathcal{V} = \{\{n\} \oplus F \mid F \text{ is finite } \& \, [\varphi_n(0)\downarrow \implies F \neq P_{\varphi_n(0)}]\}.$$

An analogue of the result of Wehner, 1999

Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be the Gödel numbering of all partially computable functions, and let $\{P_n\}_{n\in\mathbb{N}}$ be the Gödel numbering of all primitive recursive sets. Then we can define the family

$$\mathcal{V} = \{\{n\} \oplus F \mid F \text{ is finite } \& \ [\varphi_n(0)\downarrow \implies F \neq P_{\varphi_n(0)}]\}.$$

Theorem. (K, 2022). $\mathbf{DS}_{PR}(\mathcal{A}_{\mathcal{V}}) = \{f \mid f \not\leq_{PR} \emptyset\}$.

An analogue of the result of Wehner, 1999

Let $\{\varphi_n\}_{n\in\mathbb{N}}$ be the Gödel numbering of all partially computable functions, and let $\{P_n\}_{n\in\mathbb{N}}$ be the Gödel numbering of all primitive recursive sets. Then we can define the family

$$\mathcal{V} = \{\{n\} \oplus F \mid F \text{ is finite } \& \ [\varphi_n(0)\downarrow \Longrightarrow \ F \neq P_{\varphi_n(0)}]\}.$$

Theorem. (K, 2022). $\mathbf{DS}_{PR}(\mathcal{A}_{\mathcal{V}}) = \{f \mid f \not\leq_{PR} \emptyset\}$.

Theorem. (K, 2022). If $graph(g)$ is primitive recursive then there is a structure $\mathcal{A}$ such that

$$\mathbf{DS}_{PR}(\mathcal{A}) = \{f \mid f \not\leq_{PR} g\}.$$

Complements of lower cones (punctual case)

Question. Is there a structure $\mathcal{A}$ such that

$$\mathbf{DS}_{PR}(\mathcal{A}) = \{f \mid f \not\leq_{PR} \emptyset\}?$$

Question. For which functions g there is a structure $\mathcal{A}$ such that

$$\mathbf{DS}_{PR}(\mathcal{A}) = \{f \mid f \not\leq_{PR} g\}?$$

Complements of lower cones (punctual case)

Question. Is there a structure $\mathcal{A}$ such that

$$\mathbf{DS}_{PR}(\mathcal{A}) = \{f \mid f \not\leq_{PR} \emptyset\}?$$

Yes.

Question. For which functions g there is a structure $\mathcal{A}$ such that

$$\mathbf{DS}_{PR}(\mathcal{A}) = \{f \mid f \not\leq_{PR} g\}?$$

Complements of lower cones (punctual case)

Question. Is there a structure $\mathcal{A}$ such that

$$\mathbf{DS}_{PR}(\mathcal{A}) = \{f \mid f \not\leq_{PR} \emptyset\}?$$

Yes.

Question. For which functions g there is a structure $\mathcal{A}$ such that

$$\mathbf{DS}_{PR}(\mathcal{A}) = \{f \mid f \not\leq_{PR} g\}?$$

Yes, for some primitive recursively unbounded g.

Complements of lower cones (punctual case)

Question. Is there a structure $\mathcal{A}$ such that

$$\mathbf{DS}_{PR}(\mathcal{A}) = \{f \mid f \not\leq_{PR} \emptyset\}?$$

Yes.

Question. For which functions g there is a structure $\mathcal{A}$ such that

$$\mathbf{DS}_{PR}(\mathcal{A}) = \{f \mid f \not\leq_{PR} g\}?$$

Yes, for some primitive recursively unbounded g.
No, for some primitive recursively bounded g.

Unions of degree spectra

Theorem. (K, 2007) If $C_1' \equiv_T C_2' \equiv_T \emptyset'$ then the collection

$$\{X \mid X \not\leq_T C_1\} \cup \{X \mid X \not\leq_T C_2\}$$

is the degree spectrum of a structure.

Unions of degree spectra

Theorem. (K, 2007) If $C_1' \equiv_T C_2' \equiv_T \emptyset'$ then the collection

$$\{X \mid X \not\leq_T C_1\} \cup \{X \mid X \not\leq_T C_2\}$$

is the degree spectrum of a structure.

Theorem. (K, 2022) If $graph(g_1)$ and $graph(g_2)$ are primitive recursive then the collection

$$\{f \mid f \not\leq_{PR} g_1\} \cup \{f \mid f \not\leq_{PR} g_2\}$$

is the punctual degree spectrum of a structure.