Introenumerable sets and the cototal enumeration degrees

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Outline

- Give a brief review of the enumeration degrees.
- Talk about minimal subshifts as our original motivation for studying the cototal sets and degrees. Introduce uniform introenumerability.
- Talk about enumeration pointed trees and McCarthy's characterizations of cototality. Relate this to uniform introenumerability.
- Describe my recent work with Goh, Jacobsen-Grocott, and Soskova.
- Talk about the proof that there is a uniformly introenumerable set that is not of cototal degree.

The enumeration degrees

Friedberg and Rogers introduced enumeration reducibility in 1959.

Informally: $A \subseteq \omega$ is *enumeration reducible* to $B \subseteq \omega$ ($A \leqslant_e B$) if there is a uniform way to enumerate A from an enumeration of B.

Definition. $A \leqslant_e B$ if there is a c.e. set W such that

$$A = \{n \colon (\exists e)\, \langle n, e \rangle \in W \text{ and } D_e \subseteq B\},$$

where D_e is the eth finite set in a canonical enumeration.

The degree structure $\mathcal{D}_e$ induced by $\leqslant_e$ is called the *enumeration degrees*. It is an upper semi-lattice with a least element (the degree of all c.e. sets).

The total enumeration degrees

Proposition. $A \leqslant_T B \iff A \oplus \overline{A} \leqslant_e B \oplus \overline{B}$.

This suggests a natural embedding of the Turing degrees into the enumeration degrees.

Proposition. The embedding $\iota \colon \mathcal{D}_T \to \mathcal{D}_e$, defined by

$$\iota(d_T(A)) = d_e(A \oplus \overline{A}),$$

preserves the order and the least upper bound.

Definition. $A \subseteq \omega$ is *total* if $\overline{A} \leqslant_e A$ (equivalently, if $A \equiv_e A \oplus \overline{A}$). An enumeration degree is *total* if it contains a total set.

The image of the Turing degrees under the embedding ι is exactly the set of total enumeration degrees.

It is easy to prove that there are nontotal enumeration degrees. In fact, a sufficiently generic $A \subseteq \omega$ has nontotal degree.

A question from Emmanuel Jeandel

Question (Jeandel, email from Summer 2015)

If $A \leqslant_e \overline{A}$, what can be said about the enumeration degree of A?

This email inspired a paper on such enumeration degrees (Andrews, Ganchev, Kuyper, Lempp, M., A. Soskova, and M. Soskova 2019).

Definition (AGKLMSS 2019, with apologies to B. Solon)

A set $A \subseteq \omega$ is *cototal* if $A \leqslant_e \overline{A}$. An enumeration degree is *cototal* if it contains a cototal set.

Theorem (M., Soskova 2018). The cototal enumeration degrees are a dense substructure of the enumeration degrees.

Jeandel's interest in these enumeration degrees comes out of *symbolic dynamics.*

Minimal subshifts

Definition

- The *shift operator* is the map $\sigma\colon 2^\omega \to 2^\omega$ that erases the first bit of a given sequence.
- $\mathcal{C} \subseteq 2^\omega$ is a *subshift* if it is closed and shift-invariant.
- $\mathcal{C}$ is *minimal* if there is no nonempty, proper sub-subshift $\mathcal{D} \subset \mathcal{C}$.
- The *language* of subshift $\mathcal{C}$ is the set
$$L_\mathcal{C} = \{\sigma \in 2^{<\omega} \colon (\exists X \in \mathcal{C})\ \sigma \text{ is a subword of } X\}.$$

Proposition

The following are equivalent for a subshift $\mathcal{C} \subseteq 2^\omega$:

1. $\mathcal{C}$ is minimal.
2. For every $X \in \mathcal{C}$, the σ-orbit of X is dense in $\mathcal{C}$.
3. Every $X \in \mathcal{C}$ contains the same subwords (i.e., all of $L_\mathcal{C}$).

Minimal subshifts and enumeration degrees

Assume that $\mathcal{C}$ is minimal.

- Every $X \in \mathcal{C}$ can enumerate the language $L_{\mathcal{C}}$.
- Conversely, from an enumeration of $L_{\mathcal{C}}$, we can compute an element of $\mathcal{C}$.

Proposition (Jeandel). A Turing degree computes a member of a minimal subshift $\mathcal{C} \subseteq 2^\omega$ if and only if it enumerates $L_{\mathcal{C}}$.

In fact, Jeandel and Vanier (2013) proved that for a *nontrivial* minimal subshift $\mathcal{C}$, any Turing degree that computes a member of $\mathcal{C}$ also *contains* a member of $\mathcal{C}$.

> Therefore, the degrees of members of a nontrivial minimal subshift $\mathcal{C}$ are exactly the total degrees above $\deg_e(L_{\mathcal{C}})$.

$L_{\mathcal{C}}$ is cotetal and uniformly introenumerable

We are ready to explain Jeandel's email.

Proposition (Jeandel)

If $\mathcal{C}$ is a minimal subshift, then $L_{\mathcal{C}}$ is cototal (i.e., $L_{\mathcal{C}} \leqslant_e \overline{L_{\mathcal{C}}}$).

Proof Sketch.

Starting with the full tree $2^{<\omega}$, use an enumeration of $\overline{L_{\mathcal{C}}}$ to prune branches that do not extend to elements of $\mathcal{C}$.

By compactness, $\tau \in L_{\mathcal{C}}$ if and only if at some stage of this pruning process, τ is a subword of every unpruned path. $\square$

A similar compactness argument shows:

Proposition (Jeandel). If $\mathcal{C}$ is a minimal subshift, then there is an enumeration operator Γ such that $S \subseteq L_{\mathcal{C}}$ infinite $\implies L_{\mathcal{C}} = \Gamma(S)$.

We say that $L_{\mathcal{C}}$ is *uniformly introenumerable*.

Taking stock

At this point, we are left with the following questions:

1. Are the degrees of languages of minimal subshifts exactly the cototal degrees?
2. How do the uniformly introenumerable degrees (i.e., those that contain a uniformly introenumerable set) relate to the cototal degrees?

Theorem (McCarthy 2018). Every cototal enumeration degree is the degree of the language of a minimal subshift.

So all cototal degrees are uniformly introenumerable.

McCarthy's proof passes through the notion of *e-pointed trees*.

Enumeration pointed trees

Definition (Montalbán). A tree $T \subseteq 2^{<\omega}$ is *e-pointed* if it has no dead ends and every infinite path $f \in [T]$ enumerates T.

We consider several variations:

- *Baire e-pointed*: if $T \subseteq \omega^{<\omega}$.
- *uniformly e-pointed*: if every $f \in [T]$ enumerates T by a fixed operator.
- *e-pointed with dead ends*: if dead ends are allowed.

Facts

- Uniformly e-pointed trees (in $2^{<\omega}$) are cototal and uniformly introenumerable.
- If $\mathcal{C} = [T]$ is a minimal subshift, where $T \subseteq 2^{<\omega}$ has no dead ends, then T is uniformly e-pointed.

Enumeration pointed trees

Theorem (Montalbán 2021)

If a structure spectrum is the Turing-upward closure of an F_σ subset of 2^ω, then it is an *enumeration-cone* (the set of total/Turing degrees above some fixed enumeration degree).

In particular, it must be the cone above the enumeration degree of an e-pointed tree. (Furthermore, the converse holds!)

The same is true for F_σ subsets of ω^ω and Baire e-pointed trees.

Theorem (McCarthy 2018)

An enumeration degree is cototal if and only if it contains a (uniformly) e-pointed tree in $2^{<\omega}$ (possibly with dead ends).

But what about introenumerability?

Given an infinite set $I \subseteq \omega$, let $T_I \subseteq \omega^{<\omega}$ be the *tree of subsets* of I. In other words, $f \in [T_I]$ if and only if f is injective and $\text{range}(f) \subseteq I$.

Note that T_I has no dead ends.

Observation. If $I \subseteq \omega$ is (uniformly) introenumerable, then T_I is (uniformly) Baire e-pointed.

Proof. Every $f \in [T_I]$ enumerates $\text{range}(f) \geqslant_e I$ (and this is uniform if I is uniformly introenumerable). Clearly $I \geqslant_e T_I$. $\square$

So in the enumeration degrees:

cotota l $\iff$ (uniformly) e-pointed

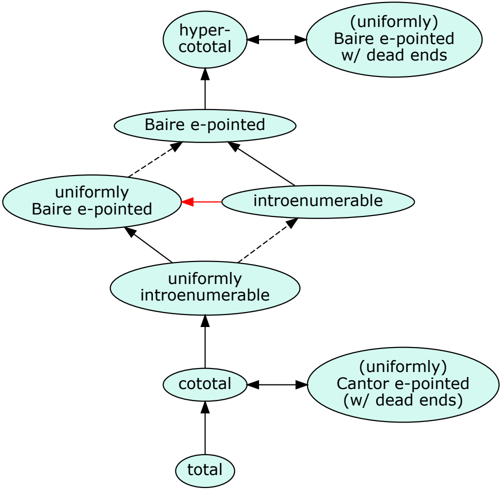
$\implies$ uniformly introenumerable $\implies$ uniformly Baire e-pointed.

These implications are strict.

Joint work with Goh, Jacobsen-Grocott, and Soskova

The solid arrows are strict.

The red arrow is open. If it is false, then the dashed arrows are also strict.



Everything else is resolved.

Hyperenumeration reducibility

Sanchis (1978) introduced *hyperenumeration reduction* ($\leqslant_{he}$) as a "higher" version of enumeration reduction.

It fits nicely into the analogy:

$$\frac{\leqslant_T}{\leqslant_h} \quad \sim \quad \frac{\text{c.e. relative to}}{\Pi^1_1 \text{ relative to}} \quad \sim \quad \frac{\leqslant_e}{\leqslant_{he}},$$

where $\leqslant_h$ is *hyperarithmetic reducibility*.

For example:

Proposition. $A \leqslant_h B \iff A \oplus \overline{A} \leqslant_{he} B \oplus \overline{B}$.

Hyper-cototality

Definition. A is called *hyper-cototal* if $A \leqslant_{he} \overline{A}$. An enumeration degree is *hyper-cototal* if it contains a hyper-cototal set. (This is equivalent to only containing hyper-cototal sets.)

Proposition (GJ-GMS)

An enumeration degree is hyper-cototal if and only if it contains a (uniformly) Baire e-pointed tree with dead ends.

Facts

- All Π^1_1 sets hyper-cototal because they are in the least he-degree.
- No 3-generic is enumeration equivalent to a Baire e-pointed tree.
- Therefore, hyper-cototal $\not\Rightarrow$ Baire e-pointed.

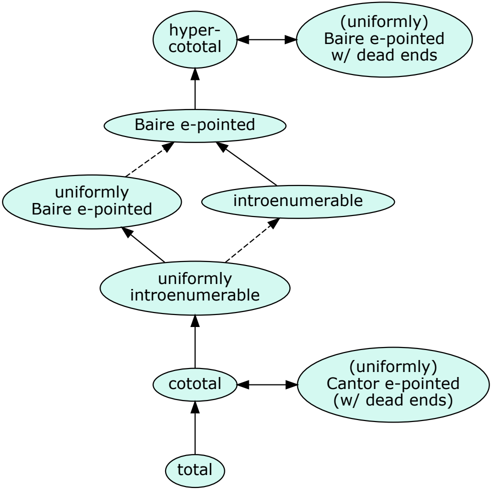
Joint work with Goh, Jacobsen-Grocott, and Soskova

Our main results are:

Thm. There is a uniformly introenumerable set that does not have cotHotal degree.

Thm. There is a uniformly Baire e-pointed tree that does not have introenumerable degree.

We discuss the first.



Uniformly introenumerable but not cototal

Theorem (Goh, Jacobsen-Grocott, M., and Soskova)

There is a uniformly introenumerable set $I \subseteq \omega$ that does not have cototal degree.

We build I by forcing.

First, assume that we have fixed a suitable enumeration operator Ψ that will witness that I is uniformly introenumerable. It must behave well with respect to finite sets.

- $\Psi(\varnothing) = \varnothing$.
- If $S \subseteq \omega$ is finite, then so is $\Psi(S)$.
- If $\Psi(S) \subseteq T$, where S and T are finite, then there is an x such that $\Psi(S \cup \{x\}) = T$.
- The previous extends (to the extent that it can) to finite sequences of pairs S_i, T_i.

The forcing notion

A forcing condition has the form $\langle G, B_k, \ldots, B_0, L\rangle$, for some $k \in \omega$, and satisfies 1–7 below.

1. $G, B_k, \ldots, B_0 \subseteq \omega$ are disjoint finite sets.
    - Every $n \in G$ is “good”; it will be in our introenumerable set.
    - Every $n \in \bigcup_{i \leqslant k} B_i$ is “bad”; we keep these out of our set.
    - Let $A = G \cup \bigcup_{i \leqslant k} B_i$.
2. $L\colon A \times \mathcal{P}(A) \to \omega \cdot 2 \cup \{\propto\}$.
3. For $C \subseteq A$, we have $(\forall n)\ L(n, C) = 0 \iff n \in \Psi(C)$.
    - $L(n, C)$ tells us how close we are to adding n to $\Psi(C)$.
    - $\propto$ will be a placeholder for finite numbers of indeterminate (but presumably large) size.
    - We order $\omega \cdot 2 \cup \{\propto\}$ by

$$0 < 1 < 2 < \cdots < \propto < \omega < \omega + 1 < \omega + 2 < \cdots .$$

The forcing notion (2)

3. For $C \subseteq A$, we have $(\forall n)\ L(n, C) = 0 \iff n \in \Psi(C)$.
4. If $C \subsetneq D \subseteq A$ and $n \in A$, then either $L(n, D) < L(n, C)$,
$$L(n, D) = \propto = L(n, C), \text{ or } L(n, D) = 0 = L(n, C).$$
   - $\propto$ allows us to sidestep the fact that $\omega \cdot 2$ is well-founded.
   - It is only allowed if n is "worse" than any element of C.
   - Let $A_j = G \cup \bigcup_{i>j} B_i$. (So $A_k = G$.)
5. If $L(n, C) = \propto$, then for some j we have $C \subseteq A_j$ and $n \in B_j$.
6. If $C \subseteq A_j$ and $n \in B_j$, then $L(n, C) \geqslant \propto$.
   - By 6 and 3, no bad number can be in $\Psi(G)$.
   - Finally, we have a transitivity property for "finiteness".
7. If $L(n, C \cup D) \leqslant \propto$ and $(\forall m \in C)\ L(m, D) \leqslant \propto$, then $L(n, D) \leqslant \propto$.

The forcing notion (3)

We say that $p' = \langle G', B'_{k'}, \ldots, B'_0, L' \rangle$ *extends* $p = \langle G, B_k, \ldots, B_0, L \rangle$, written as $p' \preceq p$, if

- $G' \supseteq G$,
- $(\forall j \leqslant k)\ B'_j = B_j$,
- $k' \geqslant k$, and
- $L' \restriction (A \times \mathcal{P}(A)) = L$.

If $\mathcal{F}$ is a filter, then let $I_{\mathcal{F}} = \bigcup_{p \in \mathcal{F}} G^p$.

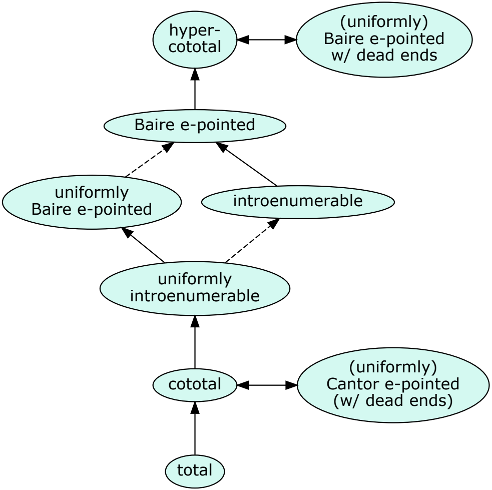
Claims. If $\mathcal{F}$ is sufficiently generic, then

- $I_{\mathcal{F}}$ is infinite. (Uses the choice of Ψ.)
- $I_{\mathcal{F}}$ uniformly introenumerable. (This is straightforward.)
- $I_{\mathcal{F}}$ does not have cototal degree. (This is where we use the sequence of bad sets.)

Why do we have a sequence of bad sets?

$$\begin{tikzcd}[column sep=small]
 & & \text{New} \\
 & & x \in G^{p_0} & & & & G^{r} \supseteq G^{q_1} \\
p & \succeq & p_0 & \succeq & p_1 \arrow[d, dashed] & \succeq & r \\
 & & p & \succeq & q_0 & \succeq & q_1 \arrow[u, dashed] \\
 & & & & x \in \bigcup B_i
\end{tikzcd}$$

Thank you!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(w/ dead ends)
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