Planar graph colouring

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Overview

- Colouring of graphs
- Recursive graphs studied for a long time (Bean, Schmerl, etc).
- All work is joint with H.T. Koh.

Theorem (Appel, Haken (1976))

Every simple planar graph (on the plane) can be coloured with at most 4 colours.

- Algorithmic content/strength of the 4 colour theorem.

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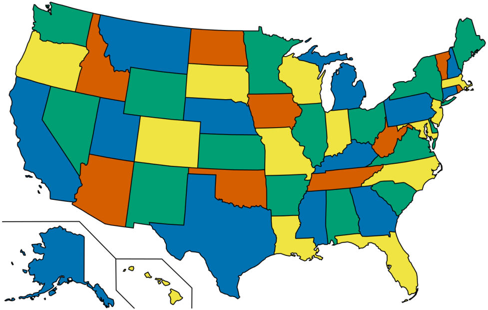
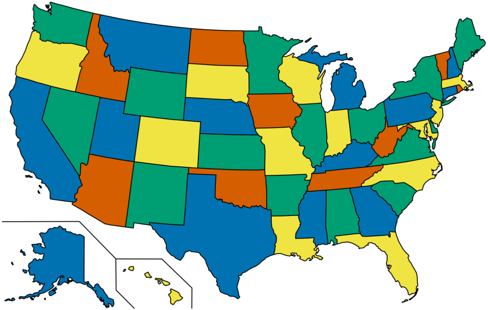
Map colouring



A four colouring of the map of the states of the US

- Problem is to colour each region so that no two contiguous regions have the same colour.
- Nevada has five neighbours, so the US map *cannot* be coloured using only three colours.

Map colouring



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The four colour theorem

- Francis Guthrie proposed this conjecture in 1852 while trying to colour the map of England.
- Kempe (1879) and Tait (1880) gave incorrect proofs, which got turned into the five colour theorem by Heawood in 1890.
- Appel and Haken finally proved the theorem in 1976, building on the computer-assisted methods developed by Heesch.
- Simplified and reproved by Robertson, Sanders, Seymour and Thomas in 1996.

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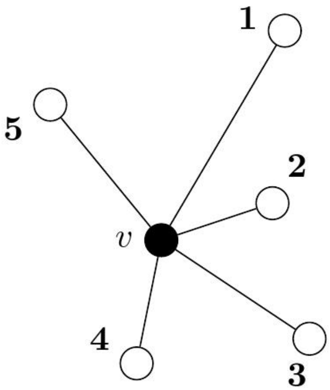
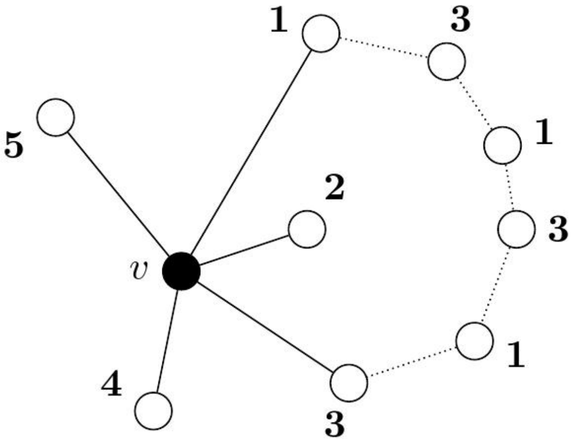
The five colour theorem

- The five colour theorem is easy to prove: By Euler characteristic, $\exists v$ such that $deg(v) \leq 5$.
- Then $G - \{v\}$ can be coloured with five colours.



The five colour theorem



Fix a 1-3 chain between the two vertices

Recursive graphs and colourings

- All graphs considered are simple, i.e. no loops and no multiple edges. (A graph in general might not be connected).
- A k-coloring of a graph (V, E) is a function $c : V \to k$ such that if $c(v) = c(v')$ then $(v, v') \notin E$.
- We of course consider infinite (countable) graphs. By compactness, Tychonoff's Theorem, etc:

Fact (De-Bruijn, Erdős)

An infinite graph is k-colorable iff every finite subgraph is k-colorable (locally k-colorable).

Recursive graphs and colourings

Theorem (Hirst)

Over RCA_0, for each $2 \leq k$, we have

$WKL_0 \Leftrightarrow$ Every locally k-colorable graph is k-colorable.

Theorem (Gasarch, Hirst)

Over RCA_0, for each $2 \leq k$, we have

$WKL_0 \Leftrightarrow$ Every locally k-colorable graph is $2k-1$-colorable.

Theorem (Schmerl)

Over RCA_0, for each $2 \leq k \leq m$, we have

$WKL_0 \Leftrightarrow$ Every locally k-colorable graph is m-colorable.

Recursive graphs and colourings

Theorem (Bean)

Every k-colorable computable graph has a low k-coloring.

- Represent vertices as nodes of a tree and edges of the tree as a possible color of the node.
- Any computable 2-colorable graph has a computable 2-coloring.

Theorem (Bean)

There is a computable 3-colorable planar graph that has no computable k-coloring for any k.

- In other words, the four colour theorem is not computably true.

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Highly recursive graphs and colourings

- A locally finite graph is highly recursive if it is computable and the degree of each vertex is computable.

Theorem (Bean)

Given each separating Π^0_1-class P and each $k \geq 3$, there is a k-colorable highly recursive graph G such that the k-colorings of G correspond to the paths of P in a degree-preserving way.

- This theorem almost establishes a relationship between WKL_0 and graph coloring principles. (For $k = 3$, G is planar).

Highly recursive graphs and colourings

Theorem (Bean)

Every highly recursive planar graph has a computable 6-coloring.

Does every highly recursive planar graph have a computable 4- or 5-coloring?

Theorem (Schmerl)

Every highly recursive k-colorable graph has a computable $2k-1$-coloring, and this result is sharp.

Theorem (Kierstead)

Every highly recursive k-colorable perfect graph has a computable $k+1$-coloring.

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Edge colourings

- (Kierstead) Every highly recursive k-edge-colorable perfect graph has a computable $k+1$-edge coloring.
- Vizing's theorem: Every k-regular graph has a $k+1$-edge colouring.
- Hence, every k-regular graph has a computable $k+2$-edge colouring.
- (Schmerl) Some computable 3-regular graph has no computable 3-edge colouring.
- (Schmerl) Is Vizing's theorem computably true?
- (Mummert, unpublished) WKL_0 is equivalent to Konig's line coloring theorem: Every bipartite graph with degree bounded by k has a k-edge-colouring.

Formalising planar graphs

- All graphs are simple. They *do not* have to be locally finite or connected, unless specified.
- All results (and definitions) are over RCA_0.
- A countable graph G is planar iff neither $K_{3,3}$ nor K_5 is a minor (or a subdivision) of G.
- (Wagner and Kuratowski-Pontryagin) For finite graphs, this is equivalent to having a plane diagram/embedding.
- Here we represent a plane diagram of a planar graph as a countable set of rational coordinates representing the coordinates of vertices, and edges. Each edge is made up of finitely many line segments.

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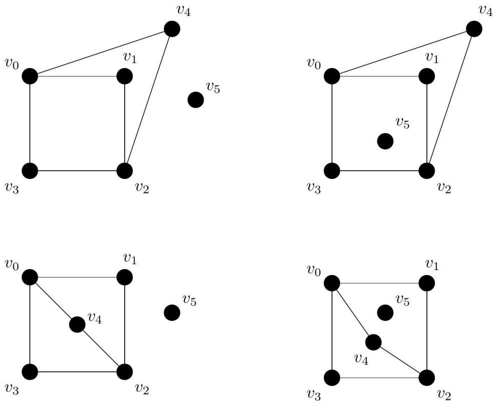
Formalising planar graphs

- Each planar graph has a plane diagram with straight line as the edges (See Wagner, Fary, Stein for finite graphs, and Thomassen for infinite graphs).
- (Erdős, see Dirac, Schuster)
  $\mathsf{WKL}_0 \vdash$ Every countable planar graph has a plane diagram.
- Is this computably true?

Plane diagrams

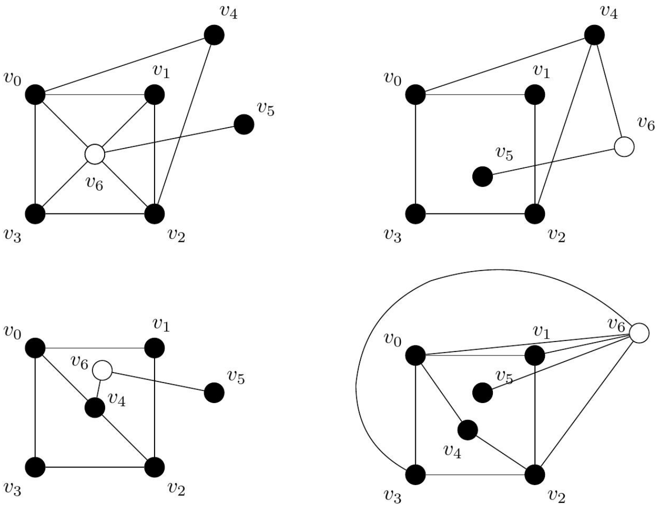
Proposition

There is a computable planar graph with no computable plane diagram.



Four possible plane drawings of the gadget

Plane diagrams



Adding the new vertex v_6 in each case

The k-color theorem

Proposition

$WKL_0 \Leftrightarrow$ Every planar graph admits a plane diagram.

- If we're working over WKL_0, we can interchangeably use the various definitions of a planar graph.

Definition

For each $k \geq 4$, define the principles:

$\mathtt{COL}(k)$: Every countable planar graph is k-colourable.

$\mathtt{COL}^*(k)$: Every countable planar graph with a computable planar diagram is k-colourable.

$\mathtt{ConnCOL}(k)$: Every countable connected planar graph is k-colourable.

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Theorem (Bean)

There is a computable 3-colorable planar graph that has no computable k-coloring for any k.

- Thus $\mathtt{COL}(k)$ is not computably true for any $k \geq 4$, but follows from WKL_0.

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WKL_0 is equivalent to each of $\mathtt{COL}(4)$, $\mathtt{COL}^*(4)$ and $\mathtt{ConnCOL}(4)$.

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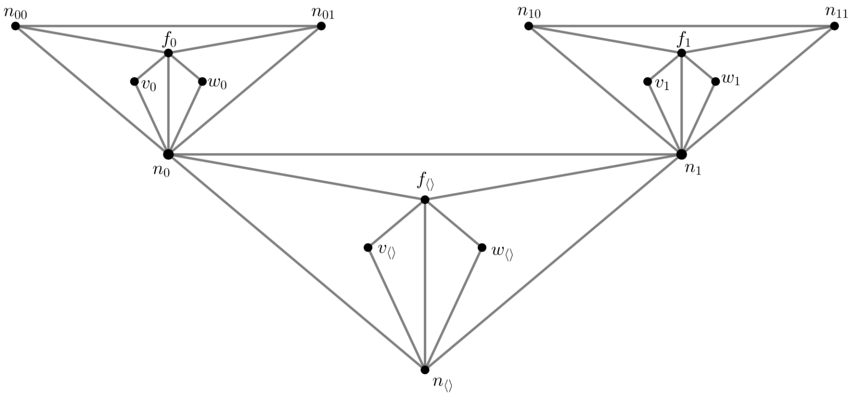
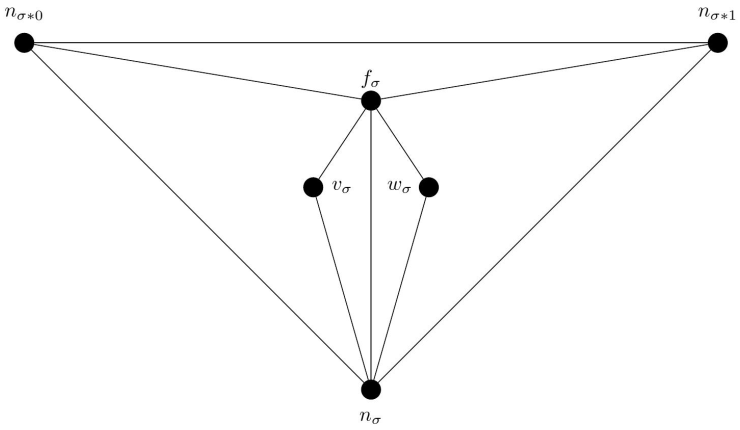
Reversing $\mathtt{COL}(4)$ to WKL_0



Given a tree T we encode $\langle\rangle, 0, 1$ into the three gadgets respectively.

Reversing $\texttt{COL}(4)$ to WKL_0

If $\sigma\texttt{0}$ dies, we add three new nodes:



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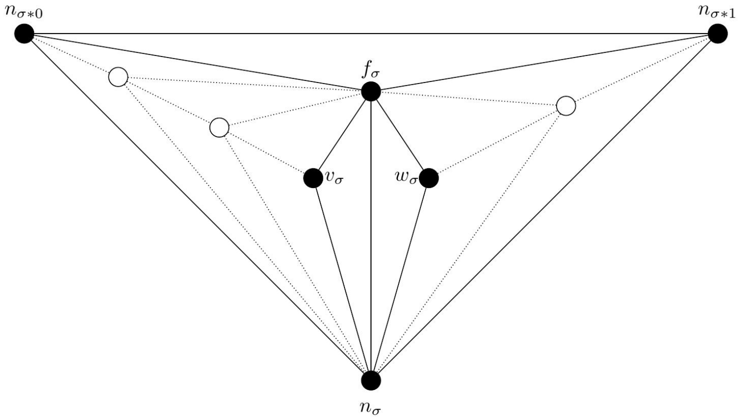

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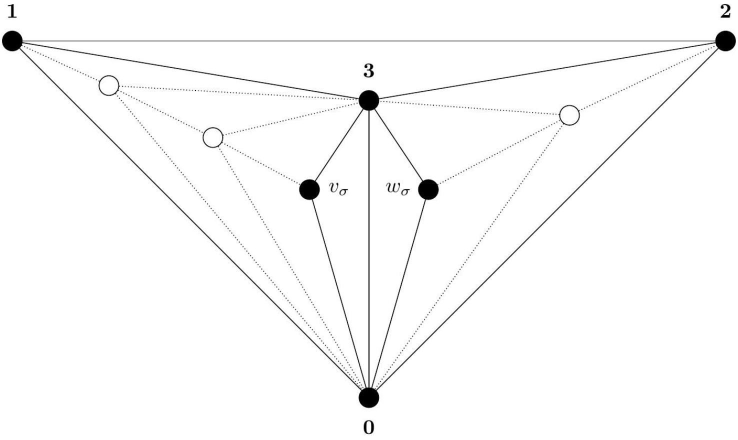

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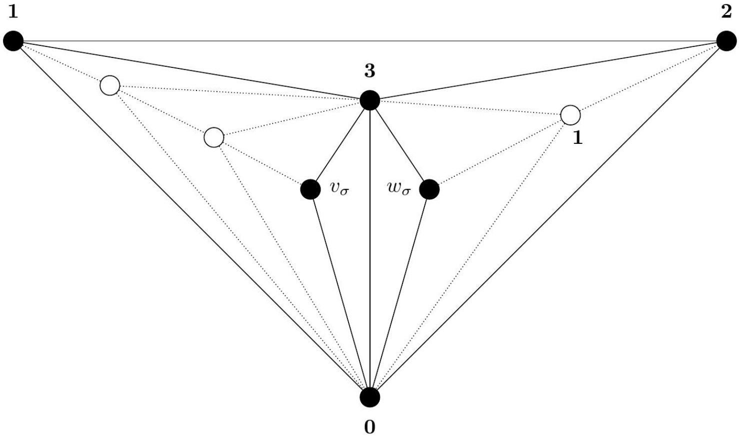

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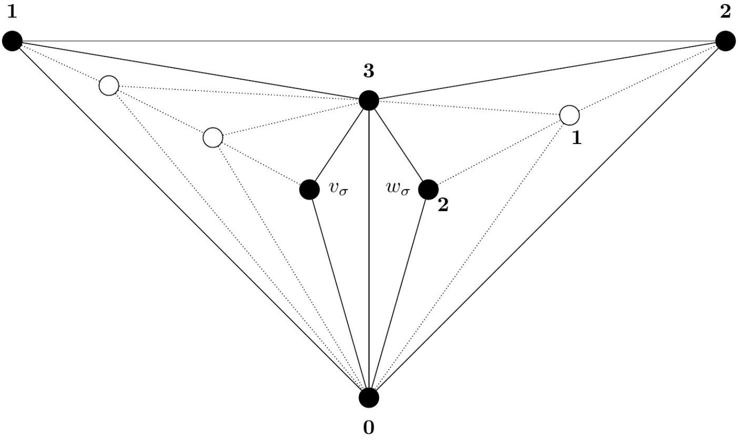

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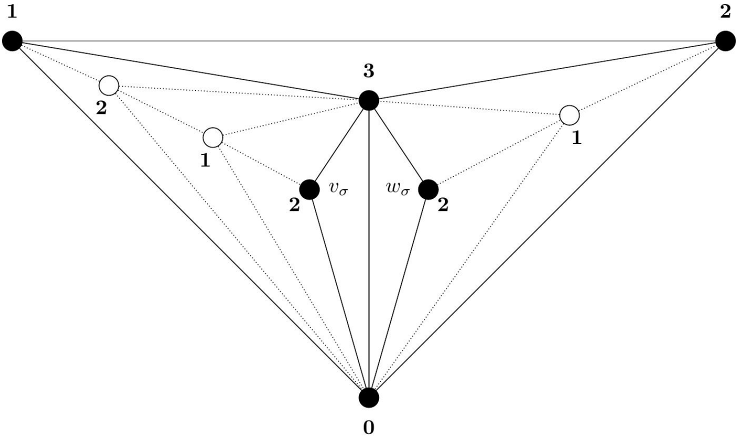

Now *any* four colouring c of the graph must satisfy

$$c(v_\sigma) = c(w_\sigma) = c(n_{\sigma 1}) \neq c(n_{\sigma 0}).$$

The k-color theorem

Theorem

For each $k \geq 4$, WKL_0 is equivalent to each of $\mathtt{COL}(k)$, $\mathtt{COL}^*(k)$ and $\mathtt{ConnCOL}(k)$.

- In the reversal $\mathsf{RCA}_0 + \mathtt{COL}(4) \vdash \mathsf{WKL}_0$, we used K_4 in our constructed graph G to force any 4-colouring of G to have little choice.
- Obviously, we can't use K_5 to show $\mathsf{RCA}_0 + \mathtt{COL}(5) \vdash \mathsf{WKL}_0$.
- $\mathsf{RCA}_0 + \mathtt{COL}(k) \vdash \mathsf{WKL}_0$ can be proved non-uniformly, and for $k \geq 7$ this is provably necessary.
- Recall that $\mathtt{DNR}(k)$: $\exists g : \omega \to \{0, 1, 2, \ldots, k-1\}$ s.t. $\forall x, g(x) \neq \varphi_x(x)$.

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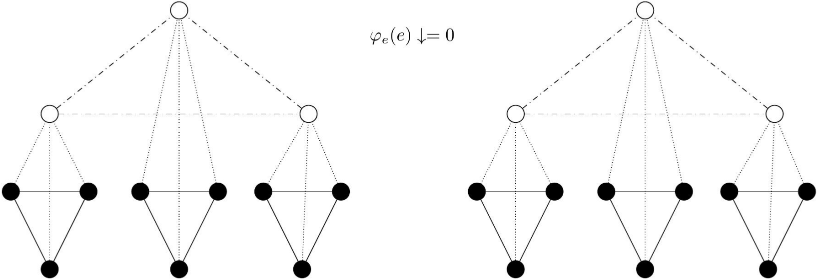
Reversing $\texttt{COL}(5)$ to $\texttt{DNR}(3)$

To encode $\varphi_e(e)$, we start with the gadget:

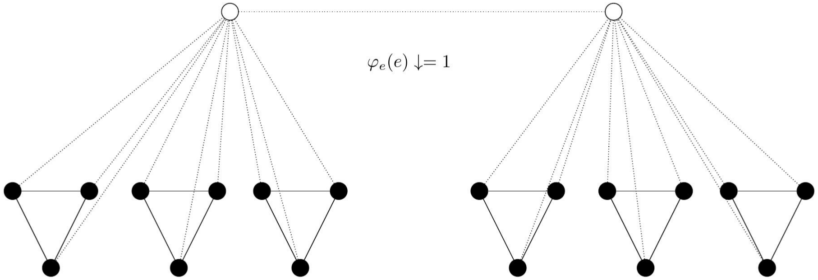
Reversing COL(5) to DNR(3)

If $\varphi_e(e) \downarrow = 0$, we add 6 new vertices and connect them to each K_3. (This diagram can be made planar).



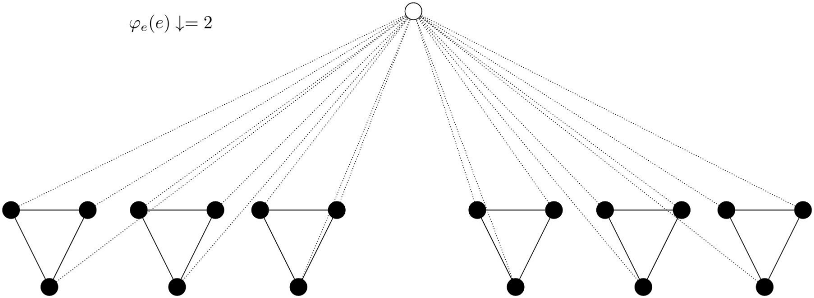
Reversing $\mathtt{COL}(5)$ to $\mathtt{DNR}(3)$

If $\varphi_e(e)\downarrow = 1$, we add 2 new vertices.



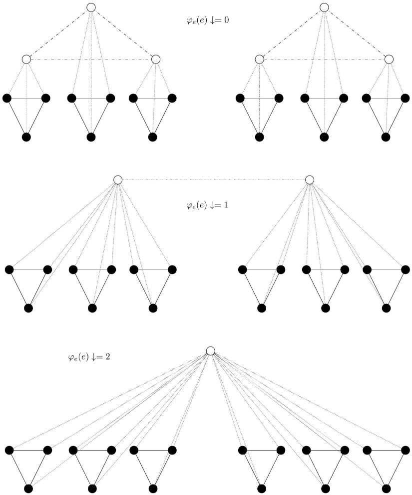
Reversing COL(5) to DNR(3)

If $\varphi_e(e)\downarrow= 2$, we add 1 new vertex and connect to all old vertices.



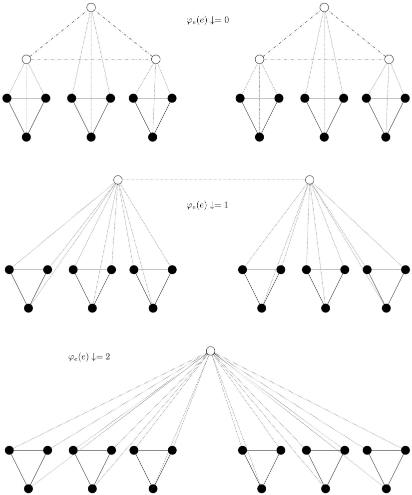
Reversing COL(5) to DNR(3)

Now given a 5-colouring of the graph, if the left set of 9 black vertices are coloured with only 3 colours, then $\varphi_e(e) \neq 0$.



Reversing COL(5) to DNR(3)

If the left and right set of 9 black vertices are coloured with 4 colours, then $\varphi_e(e) \neq 1$.



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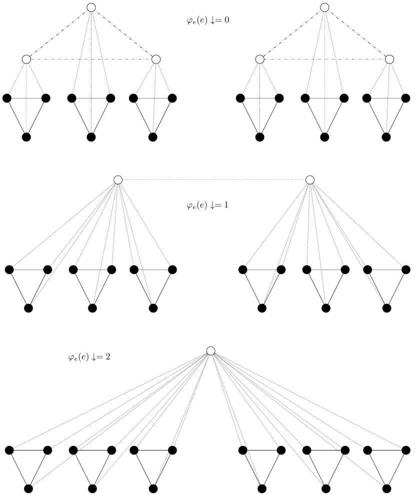
If the black vertices are coloured with all 5 colours, then $\varphi_e(e) \neq 2$.



The k-color theorem and uniformity

- This shows that $\mathsf{RCA}_0 + \mathtt{COL}(5) \vdash \mathtt{DNR}(3)$, and thus $\mathsf{WKL}_0 \leftrightarrow \mathtt{COL}(4)$ and $\mathsf{WKL}_0 \leftrightarrow \mathtt{COL}(5)$.
- To further calibrate the complexity of statements which might be equivalent in the RM sense, we use the tools from computable analysis.

Weihrauch reducibility

Definition (Dorais, Dzhafarov, Hirst, Mileti and Shafer, after Weihrauch)

Let P and Q be Π^1_2 statements of second-order arithmetic.

- $P \leq_W Q$, if $\exists \Phi, \Psi$ where Φ, Ψ are Turing reductions s.t. whenever A is an instance of P, $B = \Phi(A)$ is an instance of Q and if T is a solution to B, then $S = \Psi(T \oplus A)$ is a solution of P.
- $P \leq_{sW} Q$, if we require $S = \Psi(T)$ is a solution of P.

- If $P \leq_W Q$, then usually one can turn it into a "uniform" proof of $\mathrm{RCA}_0 \vdash Q \to P$.

The k-color theorem and uniformity

Theorem

- $\mathtt{WKL} \leq_{sW} \mathtt{COL}(4)$.
- $\mathtt{DNR}(3) \leq_{sW} \mathtt{COL}(5)$.
- $\mathtt{DNR}(4) \leq_{sW} \mathtt{COL}(6)$.
- $\mathtt{DNR}(8) \leq_{sW} \mathtt{COL}(7)$.

Obviously we should have $\mathtt{DNR}(?) \leq_{sW} \mathtt{COL}(8)$?

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- Suppose that there is a computable planar G and some Ψ such that $\Psi^c(x) \neq \varphi_x(x)$ for every x and every 8-colouring c of G.
- By the Recursion Theorem, define $\varphi_e(e) = \Psi^\sigma(e)$ where σ is a 4-colouring of a finite subgraph H of G.
- Since $G - H$ is planar, we can 4-colour $G - H$ with a different set of 4 colours, and so σ can be extended to an 8-colouring c of G.
- This is a contradiction since $\Psi^c(e) = \varphi_e(e)$.

Reversing $\mathtt{COL}(k)$ for $k \geq 8$

- $\mathtt{DNR}(k) \not\leq_W \mathtt{COL}(8)$ for any k, how can we get the reversal to WKL_0?

Definition

For $k, l \in \omega$, let $\mathtt{DNR}(k, l)$: $\exists$ an l-approximable function $g : \omega \to \{0, 1, \cdots, k-1\}$ such that $\forall x, g(x) \neq J(x)$, where $J(x)$ is universal c.e. trace with $l + 1$ many possibilities. Hence $\mathtt{DNR}(k, 0) = \mathtt{DNR}(k)$.

Theorem

- *For any* $l \geq 0$, $\mathtt{DNR}(k, l+1) \vdash \mathtt{DNR}(k) \vee \mathtt{DNR}(k, l)$.
- *For any* $n > 3$, *there are constants* k_n, l_n *such that* $\mathtt{DNR}(k_n, l_n) \leq_{sW} \mathtt{COL}(n)$.

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Corollary

Over RCA_0, for each $n > 3$, WKL_0 is equivalent to each of $\mathtt{COL}(n), \mathtt{COL}^*(n), \mathtt{ConnCOL}(n)$.

- Which of the principles $\mathtt{COL}(n), \mathtt{COL}^*(n), \mathtt{ConnCOL}(n)$ and $\mathtt{WKL}$ is uniformly obtainable from another?
- We've seen that $\mathtt{DNR} \not\leq_W \mathtt{COL}(8)$ and therefore $\mathtt{WKL} \not\leq_W \mathtt{COL}(n)$ for any $n \geq 8$.
- On the other hand, $\mathtt{WKL} \leq_{sW} \mathtt{COL}(4)$.

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$\mathtt{WKL} \not\leq_W \mathtt{COL}(7)$.

Reversing $\mathtt{COL}(k)$ for $k \geq 8$

Corollary

Over RCA_0, for each $n > 3$, WKL_0 is equivalent to each of $\mathtt{COL}(n), \mathtt{COL}^*(n), \mathtt{ConnCOL}(n)$.

- Which of the principles $\mathtt{COL}(n), \mathtt{COL}^*(n), \mathtt{ConnCOL}(n)$ and $\mathtt{WKL}$ is uniformly obtainable from another?
- We've seen that $\mathtt{DNR} \not\leq_W \mathtt{COL}(8)$ and therefore $\mathtt{WKL} \not\leq_W \mathtt{COL}(n)$ for any $n \geq 8$.
- On the other hand, $\mathtt{WKL} \leq_{sW} \mathtt{COL}(4)$.

Proposition

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Since $\texttt{DNR}(8) \leq_{sW} \texttt{COL}(7)$, the diagonalisation here must be different from what we used for $\texttt{DNR} \not\leq_W \texttt{COL}(8)$.

Lemma

It suffices to prove that for any finite planar $G \subset G_0, G_1$, there is a 7-coloring of G that extend to 7-colorings of G_0 and G_1.

Proof.

- Suppose that $\texttt{WKL} \leq_W \texttt{COL}(7)$ with some reductions Φ, Ψ.
- Take $G = \Phi(2^\omega)$, $G_0 = \Phi([0])$ and $G_1 = \Phi([1])$.
- Now fix 7-colorings $h \subset h_0, h_1$ of G, G_0, G_1 respectively.
- Wait for $\Psi(h) \supset 0$ or $\Psi(h) \supset 1$ ($\Psi(h)$ must pick a path on 2^ω).

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Lemma

It suffices to prove that for any finite planar $G \subset G_0, G_1$, there is a 7-coloring of G that extend to 7-colorings of G_0 and G_1.

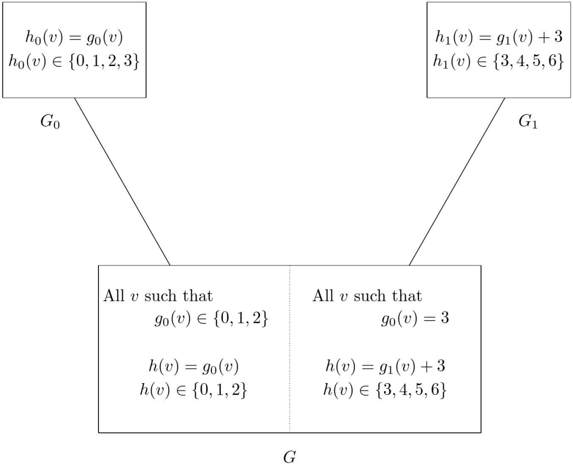
Proof.

- If $\Psi(h) \supset 0$ we remove $[0]$ from our input tree.
- Since $h_1 \supset h$, $\Psi(h_1) \supset 0$ which isn't a path on $\Psi^{-1}(G_1)$. □

Now using the lemma, we fix finite planar $G \subset G_0, G_1$.

- Fix a 4-coloring g_0 of G_0 and a 4-colouring g_1 of G_1.
- We will have $g_0 \upharpoonright G \neq g_1 \upharpoonright G$. How to define a 7-colouring h on G?

$\mathsf{WKL} \not\leq_W \mathsf{COL}(7)$



- Define h on G and h_0, h_1 on G_0, G_1 as shown.
- Clearly they are 7-colourings.

Weihrauch reductions

Theorem

Every four levels of $\mathtt{COL}(k)$ *is proper wrt* $\leq_W$*, i.e.*

$$\mathtt{COL}(4n), \mathtt{COL}(4n+1), \mathtt{COL}(4n+2), \mathtt{COL}(4n+3) \not\leq_W \mathtt{COL}(4n+4).$$

- It works because given planar graphs $G \subset \hat{G}$ (G is finite), then any k-colouring of G extends to a $k+4$-colouring of $\hat{G}$.
- We have similar extension theorems for $\mathtt{ConnCOL}$ and $\mathtt{COL}^*$:

Weihrauch reductions

Lemma

- *Given connected planar graphs $G \subset \hat{G}$ (G is finite), then any k-colouring of G extends to a $k + 3$-colouring of $\hat{G}$.*
- *Given planar graphs $G \subset \hat{G}$ (G is finite), with respective computable plane diagrams $D \subset \hat{D}$ then any $3k + 1$-colouring of G extends to a $3k + 4$-colouring of $\hat{G}$.*

Corollary

For almost every k,

- $\mathtt{COL}(k) \not\leq_W \mathtt{ConnCOL}(k)$.
- $\mathtt{COL}(k) \not\leq_W \mathtt{COL}^*(k)$.

Questions

- Does $\mathtt{WKL} \leq_{sW} \mathtt{COL}(5)$ or $\mathtt{COL}(6)$?
- Generally, are any of the reductions proper?
  $\mathtt{COL}(4n+3) \leq_{sW} \mathtt{COL}(4n+2) \leq_{sW} \mathtt{COL}(4n+1) \leq_{sW} \mathtt{COL}(4n)$.
- Calibrating the exact relationships between $\mathtt{COL}(k)$, $\mathtt{COL}^*(n)$ and $\mathtt{ConnCOL}(m)$ for various k, n, m.
- For instance, for each k, what is the least n such that $\mathtt{COL}(k) \leq_{sW} \mathtt{COL}^*(n)$ or $\mathtt{ConnCOL}(n)$?
- Principles arising from other restrictions on the planar graph, such as locally finite or highly recursive?
- Strength of Fary-Thomassen's theorem, or Grötzsch's theorem.
- Compare the colouring principles with other Weihrauch degrees, closure under parallelization, etc?

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