Two applications of the fireworks method

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1. Feeble subsets and fireworks

Nostalgia...

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This was an important first step towards proving the following conjecture: if X is **Martin-Löf random**, then there exists an infinite $Y \subseteq X$ such that Y does not compute any Martin-Löf random.

Kjos-Hanssen's argument

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Step 1: a computability-theoretic part.

Proposition

There exists a $\emptyset'$-computable 3-bushy tree $T \subseteq \omega^\omega$ such that no path of T computes a Martin-Löf random.

Kjos-Hanssen's argument

Recall: a tree is (perfectly) *k*-**bushy** if every node in the tree has at least *k* children.

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If $f : \omega \to \omega$ is a function, a tree T is (perfectly) *f*-**bushy** if any node $\sigma \in T$ has at least $f(|\sigma|)$ many children.

Kjos-Hanssen's argument

To prove the 'Step 1' theorem, one uses **bushy tree forcing**, where a forcing condition is a pair $(\tau, \mathcal{B})$ where τ is a finite bushy tree and $\mathcal{B}$ is a 'small' set of strings to be avoided, and

$$(\tau', \mathcal{B}') \leq (\tau, \mathcal{B})$$

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A very powerful tool, see *M. Khan and J.S. Miller, Forcing with bushy trees*.

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... but this where there is some wiggle room! (we'll come back to this)

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Proposition

Start with a perfect infinite ternary tree. For each node, flip a coin. Heads: remove the node; Tail: keep the node. Then with positive probability some path X of the original tree is intact (no node of X is removed).

Kjos-Hanssen's argument

This is because, in terms of survival of paths, this process is equivalent to a Galton-Watson process:

- Start wih a root node.
- This node produces 0, 1, 2 or 3 children, with respective probabilities $1/8, 3/8, 3/8, 1/8$.
- These children themselves produce 0, 1, 2 or 3 children with the same probability, etc.

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A well-known result in probability theory is that the process produces an infinite tree with positive probability when the expectation of the number of children is > 1 (which is the case here: expectation is $3/2$).

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This finishes the proof: take a $\emptyset$'-computable tree ternary T none of whose paths computes a random. Let X be Schnorr random relative to $\emptyset'$. Then $X^* \cap T$ contains a path Y which does not compute a 1-random, thus $Y \cap X$ does not either. This $X \cap Y$ is the desired subset of X.

The full result holds true

It turns out that a stronger result mentioned as a conjecture is true:

Theorem (Kjos-Hanssen and Liu, 2019)

For every 1*-random X, there is an infinite subset Y of X such that Y does not compute any* 1*-random.*

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We present an alternative proof using the fireworks machinery.

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Theorem (Kautz,1991)

Every 2-random computes a 1-generic.

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Kautz's argument generalizes easily (once reformulated as 'fireworks' by Shen and Rumyantsev) to any computable order.

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Theorem

Let $(\mathbb{P}, \leq)$ be a computable (or c.e.) order. Then every 2-random computes a decreasing sequence $p_0 > p_1 > p_2 > \dots$ such that for every c.e. subset W of $\mathbb{P}$, there is some i such that either (1) $p_i \in W$ or (2) for all $q \leq p_i$, $q \notin W$.

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(Note: In fact Demuth randomness is enough, as shown by B. and Porter).

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Theorem (B., Patey, 2014)

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Note: here 2-random is needed, Demuth random will not do because there is an extra layer of randomness on top of the fireworks argument.

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Every 2-random computes an h-perfectly bushy tree T with large h and a sequence of small c.e. sets (B_i) such that for any path X of T **that is not in any** B_i, X does not compute a 1-random.

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Moreover, h can be taken as fast-growing as we want, and the B_i as small as we want (compared to h-bushiness).

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- So $X \cap T$ must still have a path avoiding the small sets!

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The answer turns out to be **no** for these three notions, which merits a story of its own.

2. Martingales and fireworks

Probabilistic martingales

For computable randomness, for a real X, asking whether X is Y-computably random for almost all Y amounts to asking whether there exists a *probabilistic martingale d* such that with positive probability over Z:

- d^Z is total
- d^Z succeeds on X

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In that case, we just get computable randomness:

Proposition (Buss and Minnes, 2013)

The following are equivalent:

(i) *X is computably random*

(ii) *If d is a probabilistic martingale which is total with probability* 1*, then it also fails on X with probability* 1*.*

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and indeed in the general setting, we get the opposite result, in a strong sense.

Theorem (B, DR, S, 2022)

There exists a partial computably random X such that for almost every Y, X is not even Schnorr random relative to Y!

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Forcing notion: functions $f : 2^{<\omega} \to \mathbb{Q}^+$ with finite domain, $f(\emptyset) = 1$ and with the fairness condition $f(\sigma 0) + f(\sigma 1) = 2f(\sigma)$.

The order is the obvious one: $f \geq g$ if f extends g.

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The order is the obvious one: $f \geq g$ if f extends g.

This allows us to talk about *generic martingales* (they are in particular total), which can be generated via fireworks.

Building a partial computably random

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Consider an effective listing $d_1, d_2, \ldots$ of all partial computable martingales.
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- At some stage, add $(1/2) \cdot d_2$ and diagonalize against $d_1 + (1/2)d_2$ (as long as you want).

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- Continue on: after diagonalizing against $d_1 + (1/2)d_2 + \ldots + (1/2^i)d_i$, add $(1/2^{i+1})d_{i+1}$ etc.

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- Note: if at some point some d_i becomes undefined, all the better! Just remove it from the set of martingales being diagonalized against.

Probabilistic martingales, continued

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But generic martingales can win money against computable paths! (when playing long enough).

Lemma

If d is a generic martingale and R_e is a computable set, there exists $n = n(e, d)$ such that $d(R_e \restriction n_e) > n_e$.

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Probabilistic martingales, continued

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Γ induces a *measure* (not a probability measure!) ξ on generic martingales. Now, we can find a sequence (N_e) such that

$$\xi\{d \mid (\forall e)\, n(e, d) < N_e\} > 0$$

It thus suffices to use this sequence N_e as a guideline when building our partial computably random: when following a path R_e, continue until bit N_e is reached.

Probabilistic martingales, continued

Combining everything together, we thus obtain an X which is partial computably random, but such that

$$\xi\{d \mid (\exists^{\infty} e) d(X \upharpoonright N_e) > N_e\} > 0$$

Probabilistic martingales, continued

Combining everything together, we thus obtain an X which is partial computably random, but such that

$$\xi\{d \mid (\exists^\infty e) d(X \upharpoonright N_e) > N_e\} > 0$$

Thus, for a positive measure of Y, Γ^Y is a total martingale which wins against X with some linear speed, hence X is not Y-Schnorr random.

Newcomers at the zoo

So we need to add at least three new members to the randomness zoo: a.e.-Schnorr randomness, a.e.-computable randomness and a.e-partial computable randomness, where a.e.-blah-random means we are Y-blah-random with respect to almost all Y.

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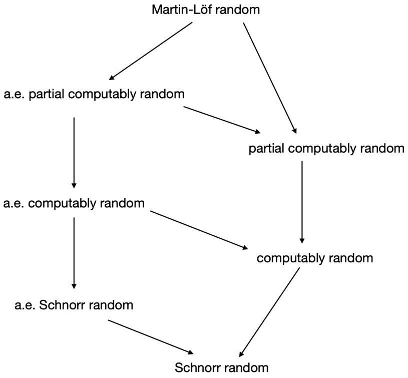
And these newcomers behave as they should...

Newcomers at the zoo



(implications are strict, no other implication)

Back to feeble subsets

So we can now state a small improvement of Kjos-Hanssen and Liu's result:

Theorem

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Theorem

If X is a.e. Schnorr random, then some infinite $Y \subseteq X$ does not compute any 1*-random.*

Which begs the question: can we replace the conclusion by 'a.e. Schnorr random'? What about Schnorr randomness?

Thank you!

Good to see you all, and hope to see you in Paris in June!