Some consequences of TD and sTD

Liang Yu

Nanjing University

March 12, 2022

TD and *sTD*

Let *AD* be the axiom of determinacy.

Definition

1. Turing determinacy (TD) says that for every set A of *Turing degrees*, either A or the complement of A contains an upper cone.
2. Strong Turing determinacy (sTD) says that for every set A of *reals* ranging Turing degrees cofinally, A has a pointed subset.

AD

Theorem (Martin)

Over ZF, $AD \to sTD \to TD$.

AD

Theorem (Martin)

Over ZF, $AD \to sTD \to TD$.

TD is more natural than AD.

Axiom of Choice

Definition

Given a nonempty set A,

1. CC_A, the countable choice for subsets of A, says that for any countable sequence $\{A_n\}_{n\in\omega}$ of nonempty subsets of A, there is a function $f:\omega\to A$ so that $\forall n(f(n)\in A_n)$.
2. DC_A, the dependent choice for subsets of A, says that for any binary relation $R\subseteq A\times A$, if $\forall x\in A\exists y\in AR(x,y)$, there is a countable sequence elements $\{x_n\}_{n\in\omega}$ so that $\forall nR(x_n,x_{n+1})$.

Determinacy v.s. Choice

Clearly AD implies $\neg AC$.

Theorem (Mycielski)

$ZF + AD$ implies $CC_{\mathbb{R}}$.

Theorem (Kechris)

$ZF + V = L(\mathbb{R}) + AD$ implies DC.

Determinacy v.s. Choice

Clearly AD implies $\neg AC$.

Theorem (Mycielski)

$ZF + AD$ implies $CC_{\mathbb{R}}$.

Theorem (Kechris)

$ZF + V = L(\mathbb{R}) + AD$ implies DC.

Question

Does $ZF + AD$ imply $DC_{\mathbb{R}}$?

TD v.s. Choice

Theorem (Peng and Y.)

$ZF + TD$ implies $CC_{\mathbb{R}}$.

TD v.s. Choice

Theorem (Peng and Y.)

$ZF + TD$ implies $CC_{\mathbb{R}}$.

Question

1. *Does $ZF + TD$ imply $DC_{\mathbb{R}}$?*
2. *Does $ZF + V = L(\mathbb{R}) + TD$ imply $DC_{\mathbb{R}}$?*

Weakly dependent choice

Definition

$wDC_{\mathbb{R}}$, weakly dependent choice, says that for any binary R over reals so that for every x, $R_x = \{y \mid R(x, y)\}$ has positive inner measure, then there is a sequence $\{X_n\}_{n\in\omega}$ of real so that $\forall n R(x_n, x_{n+1})$.

TD implies $wDC_{\mathbb{R}}$

Theorem (Peng, Wu, Y)

$ZF + TD$ implies $wDC_{\mathbb{R}}$.

Proof.

By the lowness for Schnorr randomness and $CC_{\mathbb{R}}$, there is some e so that the set
$A_e = \{x \mid$
$\Phi_e^{x''}$ computes an x-Schnorr random real; and for every real $y \leq_T$
$\Phi_e^{x''}$ and every x''-Schnorr random real r, $R(y, r)\}$
ranges cofinally.

By TD, fix such base x_0 for A_e. For every $n \geq 1$, let $r_{2n} \leq_T \Phi_e^{x_0^{(2n)}}$ be an $x_0^{(2n-2)}$-schnorr random real. Then $\forall n \geq 1 R(r_{2n}, r_{2n+2})$. □

AD v.s. Regularity

Theorem (Many set theorists, mainly due to Mycieski)

Assume $ZF + AD + DC$, every set of reals has perfect set property, and measurable, and has Baire property.

sTD v.s. Regularity

Theorem (Woodin, rediscovered by Peng, Wu and Y.)

Assume $ZF + sTD$, every set of reals has is measurable and has Baire property. If we assume $DC_{\mathbb{R}}$, then every set of reals has perfect set property.

Proof.

We only prove measurability. By $CC_{\mathbb{R}}$, it suffices to prove that if every measurable subset of A is null, then A must be null.
Otherwise, for any real x, A contains an x-Schnorr random real. Then there is some e so that the set
$B = \{x \mid \Phi_e^{x''} \in A \text{ is a Schnorr random relative to } x\}$ ranges Turing degrees cofinally. By sTD, there is a pointed subset $P \subseteq B$. Then the set $C = \{r \mid \exists x \in P(r = \Phi_e^{x''})\} \subseteq A$ is an analytic non-null set. □

Some open problems

Question (Sami)

Does $ZF + TD(+DC)$ imply the regular properties for sets of reals?

Sami proves that $ZF + TD \vdash CH$.

Some open problems

Question (Sami)

Does $ZF + TD(+DC)$ imply the regular properties for sets of reals?

Sami proves that $ZF + TD \vdash CH$.

Question

Does $ZF + TD(+DC)$ imply sTD?

Another application (1)

Theorem (Besicovitch and Davis)

For any analytic set A, $Dim_H(A) = \sup_{F\subseteq A \wedge F \text{ is closed}} Dim_H(F)$.

Theorem (Lutz and Lutz)

For any set A of reals, $Dim_H(A) = \min_x \max_{r\in A} dim_H^x(r)$.

Theorem (Slaman)

Assume that $V = L$, then BD-theorem fails for a Π^1_1-set.

One may slightly weaken the assumption to be "$(\mathbb{R})^L$ is not null".

Another application (2)

Theorem (Lempp, Miller, Ng, Turetsky and Weber)

For any real x, there is a real y low for Hausdorff dimension but $y' \geq_T x$.

Another application (3)

Theorem (Peng, Wu and Y; Crone, Fishman and Jackson proves the consequence under $ZF + DC + AD$.)

Assume that $ZF + sTD$, BD-theorem holds for every set of reals.

Proof.

Fix any nonempty set A. For the simplicity, we may assume that $Dim_H(A) = 1$.
By the results above, there is some e so that
$B = \{x \mid \Phi_e^{x'} \in A$ has effective Hausdorff dimension 1 relative to $x\}$ ranges Turing degrees cofinally. By sTD, B has a pointed subset P.
Then $C = \{r \mid \exists x \in P \Phi_e^{x'} = r\}$ is an analytic subset of A with Hausdorff dimension 1. □

More results

Theorem (Joyce and Preiss)

For any analytic set A, $Dim_P(A) = \sup_{F \subseteq A \wedge F \text{ is closed}} Dim_P(F)$.

By a similar method, one may show that Joyce-Preiss theorem holds for arbitrary set under $ZF + sTD$. Note that Slaman's result holds for the packing dimension.

Some questions

Question

1. *What is the consistency strength of BD- and JP-theorems for arbitrary sets?*
2. *What is the consistency strength that every set of Turing degrees is measurable?*

谢谢