Reverse math of Borel combinatorics

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Graph coloring

Fact. (Folklore)

(a) Every graph with no odd cycles has a 2-coloring.

(b) There is a Borel graph with no odd cycles and no Borel 2-coloring.

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(a) Every graph with no odd cycles has a 2-coloring.

(b) There is a Borel graph with no odd cycles and no Borel 2-coloring.

Proof of (b). Take the unit circle $\mathbb{T}$ with an irrational rotation $S : \mathbb{T} \to \mathbb{T}$.

- For $x, y \in \mathbb{T}$, put an edge between if $S(x) = y$ or $S(y) = x$.
- Any Borel coloring of $\mathbb{T}$ is measurable.
- Any measurable coloring would have an interval I that is almost monochromatic (at least 99% of one color)
- There is an odd k such that $S^{(k)}(I) \cap I$ is large.
- Some $x \in I$ has the same color as $S^{(k)}(x)$, contradiction.

The same can be proved by Baire category.

Hall's Theorem

If G is a graph, a perfect matching is a subset P of the edges of G such that each vertex of G is the endpoint of exactly one edge in P.

- (Hall 1935) Every n-regular graph with no odd cycles has a perfect matching.
- (Marks 2016) For $n \geq 2$ there is a Borel n-regular acyclic graph with no Borel perfect matching. (Proof used Borel determinacy.)
- (Kun 2021) There is a 3-regular acyclic Borel graph with no measurable perfect matching.
- (Conley & Miller 2017) For $n \geq 3$, every n-regular acyclic Borel graph has a perfect matching with the property of Baire.

Question: Is there any way to prove Marks theorem via Baire category?

Brooks' theorem

Let $n \geq 3$.

- (Brooks 1941) If G is a graph where each vertex has degree at most n but has no n-clique, then there is an n-coloring of G.
- (Marks 2016) There is a Borel n-regular acyclic graph with no Borel n-coloring. (Proof uses Borel Determinacy.)
- (Conley, Marks, Tucker-Drob 2016) Every Borel n-regular acyclic graph has a measurable n-coloring and an n-coloring with the property of Baire.

Question: Is there any way to prove Marks theorem via measure or category?

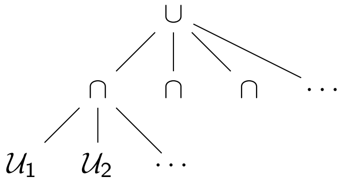
Reverse Mathematics

"When the theorem is proved from the right axioms, the axioms can be proved from the theorem." (Friedman 1968)

- Suppose Axiom A is used to prove Theorem T.
- Fix a base theory, some little axioms strong enough so T makes sense, but weaker than A.
- If T and the base theory together imply A, then A is necessary for proving T.
- To show that some other axiom B *cannot* be used to prove T, we build a model of "mathematics" in which B is true and T is false.

Second order arithmetic

- Most math can be carried out in second order arithmetic (SOA).
- In SOA, there are two kinds of objects, natural numbers and subsets of natural numbers (infinite bit sequences)
- Everything else is coded. For example, a Borel set B is given by a (code for a) well-founded, countably branching $\cap/\cup$/clopen-labeled tree describing how to make B.



$\Longrightarrow$ 00001001110111101...

- The axioms of SOA, including the axioms of Peano Arithmetic for the natural numbers and various set existence axioms, suffice for most mathematics outside set theory.

Borel set membership

Suppose we have a Borel set B and want to know if $X \in B$.
(B is coded by a $\cap/\cup/$clopen-labeled tree $S \subseteq \omega^{<\omega}$)

There is an inductive "procedure":

$$X \in B \iff \begin{cases} X \in B & \text{if } B \text{ is a basic open set or its complement} \\ \exists n[X \in B_n] & \text{if } B = \bigcup_n B_n \\ \forall n[X \in B_n] & \text{if } B = \bigcap_n B_n. \end{cases}$$

One step is arithmetic, and the recursion has transfinite depth.

The axiom of Arithmetic Transfinite Recursion (ATR_0) roughly states that a procedure such as the above has a well-defined output, namely an *evaluation map* $f : S \to \{0, 1\}$ which indicates X's membership status in all subtrees of S.

Consequences of ATR_0

Some consequences of ATR_0

- Evaluation maps always exist.
- Every Borel set is measurable.
- Every Borel set has the property of Baire.

However, Borel determinacy does not even hold in SOA.

Borel sets without ATR_0

Definition. (ADMSW 2020) A Borel set coded by S is *completely determined* (c.d.) if every $X \in 2^\omega$ has an evaluation map in S.

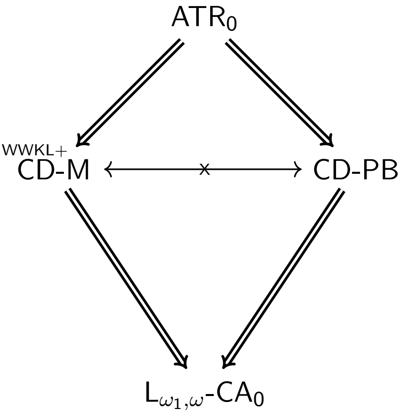
Definition. A formula ϕ of $L_{\omega_1,\omega}$ is *completely determined* if there is a function f : Subformulas(ϕ) $\to \{T, F\}$ which evaluates the formula. $\mathrm{L}_{\omega_1,\omega}$-$\mathrm{CA}_0$ states: for every sequence $\langle \phi_n \rangle$ of c.d. formulas of $L_{\omega_1,\omega}$, the sequence $\langle f_n \rangle$ of evaluation maps exists.

Prop. Over $\mathrm{L}_{\omega_1,\omega}$-$\mathrm{CA}_0$: complements, countable unions, countable intersections, and continuous pre-images of c.d. Borel sets are c.d. Borel.

Definition

- Let CD-PB be the principle
  "Every c.d Borel set has the property of Baire."
- Let CD-M be the principle
  "Every c.d. Borel set is measurable."

Relation of principles over RCA_0



Distinguishing the axioms

Fact. Every ω-model of CD-PB or CD-M is closed under hyperarithmetic reduction.

Theorem. (ADMSW '20, W '21) Both CD-PB and CD-M are strictly weaker than ATR_0.

Prop. Neither CD-PB nor CD-M implies the other.
Thus, "no Borel 2-coloring of $\mathbb{T}$" cannot prove either one.

Questions for $n \geq 3$.

- Can CD-PB prove that there is a Borel n-regular graph with no odd cycles and no perfect matching?
- Can either CD-PB or CD-M prove there is an acyclic Borel n-regular graph with no Borel n-coloring?

Σ^1_1-generics and Π^1_1-randoms

Background from HN07, CNY08, GM17, Stern 1973/5.

Definition (Technically theorems)

- A real G is Σ^1_1-generic if and only if G is Δ^1_1-generic and $\omega_1^G = \omega_1^{ck}$.
- A real R is Π^1_1-random if and only if R is Δ^1_1-random and $\omega_1^R = \omega_1^{ck}$.

Let $X^{[<k]}, X^{[k]}, X^{[\neq k]}$ denote the first k columns of X, the kth column of X, all columns but the kth column of X.

Van Lambalgen's Theorem

- If G is Σ^1_1-generic, $G^{[k]}$ is Σ^1_1-generic relative to $G^{[\neq k]}$.
- If R is Π^1_1-random, $R^{[k]}$ is Π^1_1-random relative to $G^{[\neq k]}$.

Models of CD-PB and CD-M

We have the following ω-models (from ADMSW'20, W'21)

- Let G be Σ^1_1-generic

$$\mathcal{M}_G = \bigcup_{k<\omega} HYP(G^{[<k]})$$

  Then $\mathcal{M}_G \models$ CD-PB $+ \neg$CD-M.

- Let R be Π^1_1-random

$$\mathcal{M}_R = \bigcup_{k<\omega} HYP(R^{[<k]})$$

  Then $\mathcal{M}_R \models$ CD-M $+ \neg$CD-PB.

- Any model of CD-PB must contain Δ^1_1-generics, and any model of CD-M must contain Δ^1_1-randoms. So neither principle holds in HYP.

Borel sets in HYP

Theorem (Towsner, Weisshaar & W.) In *HYP*, if G is a c.d. Borel n-regular graph with no odd cycles, then

- G has a c.d. Borel 2-coloring
- G has a c.d. Borel perfect matching

Of course, the "Borel" 2-coloring and perfect matching are given by pseudo-Borel codes:

- truly ill-founded
- but *HYP* believes well-founded and c.d.

This shows that $\mathsf{L}_{\omega_1,\omega}$-$\mathsf{CA}_0$ is a suitable base theory for exploring the strength of these theorems.

α-recursion theory

Let α be any admissible ordinal (e.g. ω_1^{ck}, the least uncomputable ordinal)

Consider the initial segment L_α of Gödel's constructible universe L.

A subset $A \subseteq L_\alpha$ is called *α-c.e.* if A is $\Sigma_1(L_\alpha)$. That is, there is a Σ_1 formula ϕ in the language of set theory such that

$$x \in A \iff L_\alpha \models \phi(x)$$

An α-c.e. set can be understood as the result of a meta-computation of length α because

$$L_\alpha \models \phi(x) \iff (\exists \beta < \alpha) L_\beta \models \phi(x).$$

A subset $A \subseteq L_\alpha$ is called *α-computable* if A is $\Delta_1(L_\alpha)$.

Characterization of the Borel subsets according to HYP

Recall that $L_{\omega_1^{ck}} \cap 2^\omega = HYP$.

The statements $\exists f[X \in_f B]$ and $\exists f[X \notin_f B]$ are each $\Sigma_1(L_{\omega_1^{ck}})$.

So $X \in B$ is $\Delta_1(L_{\omega_1^{ck}})$.

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So $X \in B$ is $\Delta_1(L_{\omega_1^{ck}})$.

Theorem. (Towsner, Weisshaar, W.) For any $A \subseteq HYP$, TFAE.

- There is a completely determined Borel code for A in HYP.
- There is a determined Borel code for A in HYP.
- A is ω_1^{ck}-computable.

ω_1^{ck}-computability

Recall: A set A is ω_1^{ck}-computable iff there are Σ_1 formulas ϕ, ψ such that

- $x \in A \iff (\exists \beta < \omega_1^{ck}) L_\beta \models \phi(x)$
- $x \notin A \iff (\exists \beta < \omega_1^{ck}) L_\beta \models \psi(x)$

Fact. Uniformly in β, $\emptyset^{(\omega \cdot \beta)}$ computes a model of L_β.

Thus, $A \subseteq HYP$ is ω_1^{ck}-computable if and only if there is a procedure Γ such that

- For all $x \in HYP$, there is $\beta < \omega_1^{ck}$ s.t. $\Gamma(x^{(\beta)})$ converges, and
- For all $x \in HYP$,

$$x \in A \iff (\exists \beta < \omega_1^{ck}) \Gamma(x^{(\beta)}) = 1.$$

First example

Theorem (TWW). In *HYP*, there is a completely determined Borel well-ordering of the reals.

Proof. For any $x \in HYP$, let α_x be the least ordinal such that

$$x \leq_T \emptyset^{(\alpha_x)}$$

and let e_x be the least number such that

$$x = \Phi_{e_x}^{\emptyset^{(\alpha_x)}}.$$

The ordering we desire is

$$x < y \iff \alpha_x < \alpha_y \text{ OR } (\alpha_x = \alpha_y \text{ and } e_x < e_y)$$

This ordering is clearly ω_1^{ck}-computable.
(We can tell whether $x < y$ uniformly in $(x \oplus y)^{(\alpha_x + \alpha_y + 2)}$)

Diagonalizing against Borel colorings

Theorem. (Marks '16) For every $n \geq 2$ there is a Borel n-regular acyclic graph with no Borel n-coloring.

Prop (TWW) In *HYP*,

(a) every n-regular Borel graph with no odd cycles has a Borel 2-coloring.
(b) there is an acyclic Borel graph such that every vertex has degree at most 2, but this graph has no Borel 2-coloring.

Proof of (b).

- To defeat the eth ω_1^{ck}-computable coloring Γ_e,
- At stage 0, set out two vertices which are connected to nothing.
- If Γ_e ever colors both, connect them in an even or odd length chain to make the coloring wrong.

What regularity gives us

Lemma. In *HYP*, suppose that G is a Borel n-regular graph. Then for each vertex x, there is $\alpha < \omega_1^{ck}$ such that $x^{(\alpha)}$ computes every vertex in the connected component of x, and all eval-maps for edges in the component.

Proof.

- For each $k \in \omega$, there are finitely many $y \in HYP$ such that y is within graph-distance k of x.
- The join of these y and the eval-maps of all associated edges is therefore hyperarithmetic.
- *By n-regularity*, for the following statement is arithmetic: $P(w, k)$: w is a finite join of exactly the $\leq k$-distant y and eval-maps.
- For each k, let α_k be least s.t. $x^{(\alpha_k)}$ computes such w.
- Apply Σ_1^1-bounding.

Using regularity to 2-color

Theorem. (Marks '16) For every $n \geq 2$ there is a Borel n-regular acyclic graph with no Borel n-coloring.

Prop. (TWW) In *HYP*,

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(b) there is an acyclic Borel graph such that every vertex has degree at most 2, but this graph has no Borel 2-coloring.

Proof of (a).

- Given x, wait until a stage α at which $x^{(\alpha)}$ computes all elements of its connected component plus evaluation maps.
- Let y be the *HYP*-least element of the connected component.
- Find a path between x and y.
- Color x according to path length parity.

Perfect matching

Prop. (TWW) In *HYP*, every Borel n-regular graph with no odd cycles has a Borel perfect matching.

Proof.

- Given vertices x, y, wait until a stage α at which $x^{(\alpha)}$ computes all elements of its connected component plus evaluation maps.
- Let w be the *HYP*-least list of the vertices and edges of the component.
- The set of perfect matchings on the component is $\Pi^0_1(w)$.
- Include (x, y) iff this edge appears in the leftmost perfect matching.

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Question: Is there any way to prove Marks theorem via Baire category?

Generalizing to $\mathcal{M}_G$

Question: For $n \geq 3$, in $\mathcal{M}_G$, does every n-regular acyclic Borel graph have a Borel perfect matching?

- Roughly speaking, in $\mathcal{M}_G$, we can still run ω_1^{ck}-algorithms on inputs x and form c.d. Borel sets which summarize the results.
- However, we lost the ordering ($\mathcal{M}_G \models$ CD-PB and there is no well-ordering of the reals with the property of Baire).
- We also lost the general assurance that the whole connected component of x is $\Delta_1^1(x)$ (though this remains true for all the graphs people actually use)

Future Directions

- Is there a Borel combinatorial zoo below ATR_0? Details?
- Are there any theorems of ordinary math or Borel combinatorics equivalent to CD-PB or CD-M?
- Is there another regularity property of Borel sets which suffices to ensure those theorems about Borel sets which hold by either measure or category arguments?
- What is the reverse math strength of "There is a Borel d-regular acyclic graph with no Borel d-coloring" for $d \geq 3$?

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Method of decorating trees

Setup: Suppose $\mathcal{M}$ an ω-model that is hyperarithmetically closed and has pseudo-ordinals.

Suppose P_α, N_α are sets of Borel rank $\sim \alpha$ that are pairwise disjoint and

$$\mathcal{M} \subseteq \bigcup_{\alpha \in \mathrm{Ord} \cap M} P_\alpha \cup N_\alpha$$

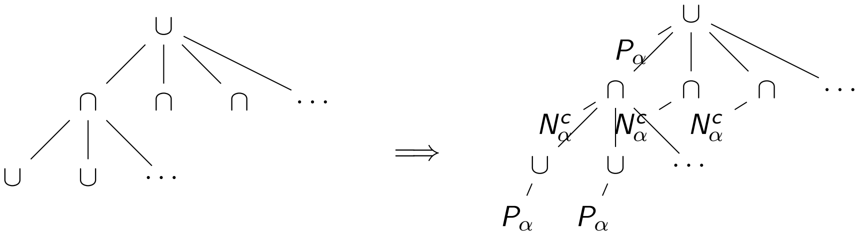
For example, if A is $\Delta_1(L_{\omega_1^{ck}})$, then we could have

- $P_\alpha = \{X : X \in A \text{ and this is first witnessed by } L_\alpha\}$
- $N_\alpha = \{X : X \notin A \text{ and this is first witnessed by } L_\alpha\}$

Claim: in $\mathcal{M}$ there is a completely determined Borel code for

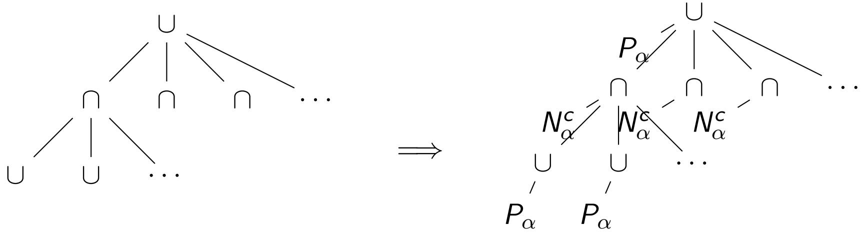
$$\mathcal{M} \cap \bigcup_{\alpha \in \mathrm{Ord} \cap \mathcal{M}} P_\alpha$$

Now, starting with an ill-founded tree T of rank α^*, for all $\alpha < \alpha^*$ we will decorate it with Borel codes for P_α and N_α as follows:



T Decorated T

Only add P_α and N_α to nodes of rank larger than these decorations. In this way the rank of T is not increased.



T Decorated T

This decorated T is completely determined on elements $X \in P_\alpha \cup N_\alpha$.
The evaluation map f can be computed in about α jumps of X as follows.

- On nodes of rank $<\approx \alpha$, use $X^{(\alpha)}$ directly to compute f
- On nodes of rank $\geq\approx \alpha$, f is constant 0 or 1 depending on if X is in P_α or N_α.

Decorating trees

Problem: If we decorate with P_1 and then with P_α, we lost the benefit of decorating with P_1

Solution: Also decorate the decorations.

This results in a tree T which $\mathcal{M}$ believes is a CD-Borel code for $\bigcup_\alpha P_\alpha$.