Equivalence Relations on Reals, and Learning for Algebraic Structures

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Learning for families of algebraic structures

- Fix a computable signature L. Let $\mathcal{K}$ be a countable family of countable L-structures.
- Step-by-step, we obtain larger and larger finite pieces of an L-structure $\mathcal{S}$.
  In addition, we assume that this $\mathcal{S}$ is isomorphic to some structure from the class $\mathcal{K}$.

Problem

Is it possible to identify (in the limit) the isomorphism type of the structure $\mathcal{S}$?

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In a sense, the problem combines the approaches of algorithmic learning theory and computable structure theory:

We want to learn the family $\mathcal{K}$ up to isomorphism.

Informal examples

Example 1. Consider two undirected graphs G_1 and G_2:

- G_1 has infinitely many cycles of size 3, and nothing else;
- G_2 has infinitely many cycles of size 4, and nothing else.

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One can learn the family $\mathcal{K} = \{G_1, G_2\}$ via the following effective procedure:

- Wait until the input graph $\mathcal{S}$ shows a cycle of size 3 or 4.
- When (the first) such cycle appears in the input, start (forever) outputting the natural guess:
  "$\mathcal{S} \cong G_1$" for size 3, or
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Remark

If $\mathcal{S}$ is *not isomorphic* to a structure from $\mathcal{K}$, then we do not care about the behavior of the learning procedure on $\mathcal{S}$.

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Observation

The graphs G_1 and G_2 are "separated" by $\exists$-sentences (inside the class $\mathcal{K}$):

$$(G_1 \text{ has a cycle of size } 3), \text{ and } (G_2 \text{ has a cycle of size } 4).$$

Informal examples

Roughly speaking, the (classical) algorithmic learning is a Δ_2^0 process (i.e., limit computable).

Example 2. The pair of linear orders $\{\omega, \omega^*\}$ can be learned.
Note that they are "separated" by $\exists\forall$-sentences:

(ω has a least element), and (ω^* has a greatest element).

Informal examples

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Learning procedure: Our input structure $\mathcal{S}$ (which is isomorphic either to ω or to ω^*) is given in stages.

At a stage s of the learning process, we find:

- ℓ_s is the $\leq_{\mathcal{L}}$-least element in the current *finite* linear order $\mathcal{L}_s$;
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Then we ask:

- For how many previous stages our ℓ_s has been the least element? More formally, this is given by the counter $c[\ell_s] = \max\{t \leq s : \ell_{s-t} = \ell_s\}$.
- For how many stages our r_s has been the greatest element? This is given by another counter: $c[r_s] = \max\{t \leq s : r_{s-t} = r_t\}$.

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After that, our conjecture at the stage s is straightforward:

- if $c[\ell_s] > c[r_s]$, then output "$\mathcal{S} \cong \omega$";
- otherwise, output "$\mathcal{S} \cong \omega^*$".

The formal learning paradigm

Fix a computable relational signature L. For convenience, we will consider only L-structures $\mathcal{S}$ with domain ω.

We fix some Gödel encoding, and we identify L-structures with elements of the Cantor space 2^ω.

Consider a family of L-structures $\mathcal{K} = \{\mathcal{A}_i : i \in \omega\}$. Here we assume that the structures $\mathcal{A}_i$, $i \in \omega$, are pairwise not isomorphic.

We need to specify four things:

1. The learning domain.
2. The hypothesis space.
3. What is a learner?
4. When a learning process is successful?

The discussed learning paradigm appears in:

- Martin and Osherson 1998;
- Fokina, Kötzing, and San Mauro 2019;
- B., Fokina, and San Mauro 2020.

The components of our learning paradigm

(1) The *learning domain*

$$\mathrm{LD}(\mathcal{K}) = \{\mathcal{S} : \mathcal{S} \cong \mathcal{A}_i \text{ for some } i \in \omega, \text{ and } \mathrm{dom}(\mathcal{S}) = \omega\}.$$

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(2) The *hypothesis space* $\mathrm{HS}(\mathcal{K}) = \omega \cup \{?\}$.

(3) A *learner* M sees (stage by stage) finite pieces of data about a given structure from $\mathrm{LD}(\mathcal{K})$, and M outputs conjectures from $\mathrm{HS}(\mathcal{K})$. More formally,

$$M \text{ is a function from } 2^{<\omega} \text{ to } \mathrm{HS}(\mathcal{K}).$$

If $M(\sigma) = i$, then this means: "the finite piece σ looks like an isomorphic copy of $\mathcal{A}_i$".

If $M(\sigma) = ?$, then this means that M abstains from giving a meaningful conjecture.

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(4) The learning is *successful* if:
for every $\mathcal{S} \in \mathrm{LD}(\mathcal{K})$, if $\mathcal{S}$ is an isomorphic copy of $\mathcal{A}_i$, then

$$\lim_{k\to\infty} M(\mathcal{S} \upharpoonright k) = i.$$

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The family $\mathcal{K}$ is <u>learnable</u> (up to isomorphism) if there exists a learner M that successfully learns the family $\mathcal{K}$.

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[More formally, one should say that $\mathcal{K}$ is $\mathbf{InfEx}_{\cong}$-learnable:

- **Inf** means learning from informant: a learner M obtains both positive and negative data about a structure;
- **Ex** means "explanatory": this is about the particular success criterion.]

A syntactic characterization

Theorem (B., Fokina, and San Mauro 2020)

Let $\mathcal{K} = \{\mathcal{A}_i : i \in \omega\}$ be a family of countable L-structures. Then the following conditions are equivalent:

(i) The family $\mathcal{K}$ is learnable.

(ii) There are Σ_2^{inf} sentences ψ_i, $i \in \omega$, such that

$$\mathcal{A}_i \models \psi_j \text{ if and only if } i = j.$$

In other words, inside the class $\mathcal{K}$, each $\mathcal{A}_i$ is distinguished by its own Σ_2^{inf} sentence ψ_i.

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This gives a useful tool for studying learning in familiar algebraic classes. For example, using the results of [Montalbán 2010], we obtain:

Theorem (B., Fokina, and San Mauro 2020)

There are no learnable infinite families of linear orders.

One interesting further direction is the following:

Problem

What happens if we change the hypothesis space $\mathrm{HS}(\mathcal{K})$?

For example, one can require that:

- our list of structures $(\mathcal{A}_i)_{i\in\omega}$ *should be* uniformly computable, but
- the list *could* contain repetitions (i.e., it could be that $\mathcal{A}_i \cong \mathcal{A}_j$ for $i \neq j$).

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Not much is known in this direction.

Lemma (B. and San Mauro 2021)

If $\mathcal{K}$ is a finite learnable family of computable L-structures, then $\mathcal{K}$ is learnable by a $\mathbf{0}'$-computable learner. This fact does not depend on the arrangement of $\mathrm{HS}(\mathcal{K})$.

Learning families of structures
with the help of Borel equivalence relations.

Joint work with V. Cipriani and L. San Mauro.

The syntactic characterization, revisited

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The key ingredient in the proof of (i)⇒(ii) is the (relativized) Pullback Theorem of Knight, S. Miller, and Vanden Boom (2007). The Pullback Theorem talks about *Turing computable embeddings* (tc-embeddings).

Let $\mathcal{K}_0$ be a class of L_0-structures, and $\mathcal{K}_1$ be a class of L_1-structures. A Turing operator Φ is a *tc-embedding* from $\mathcal{K}_0$ into $\mathcal{K}_1$ if:

- for every $\mathcal{A} \in \mathcal{K}_0$, $\Phi^{\mathcal{A}}$ is a structure from $\mathcal{K}_1$;
- for all $\mathcal{A}, \mathcal{B} \in \mathcal{K}_0$,

$$\mathcal{A} \cong \mathcal{B} \;\Leftrightarrow\; \Phi^{\mathcal{A}} \cong \Phi^{\mathcal{B}}.$$

Here the reductions emerge

Let X and Y be non-empty sets. Let E be an equivalence relation on X, and let F be an equivalence relation on Y. A function g is a *reduction* from E to F if for all $x, y \in X$, we have

$$(x\,E\,y) \;\Leftrightarrow\; (g(x)\,F\,g(y)).$$

In descriptive set theory:

One takes X and Y as Polish spaces. If the function g is Borel, then g is a Borel reduction.

If the function g is continuous, then g is a continuous reduction.

A descriptive set-theoretic characterization of learning

One of the benchmark equivalence relations on 2^ω is the relation E_0:

$$(\alpha \, E_0 \, \beta) \;\Leftrightarrow\; (\exists n)(\forall m \geq n)(\alpha(m) = \beta(m)).$$

Recall that $\mathrm{LD}(\mathcal{K}) = \{\mathcal{S} : \mathcal{S} \cong \mathcal{A}_i \text{ for some } i \in \omega\} \subseteq 2^\omega$.

Theorem 1 (B., Cipriani, San Mauro)

Let $\mathcal{K} = \{\mathcal{A}_i : i \in \omega\}$ be a family of countable L-structures. Then the following conditions are equivalent:

(a) The family $\mathcal{K}$ is learnable.

(b) There is a continuous function $\Gamma \colon 2^\omega \to 2^\omega$ such that for all $\mathcal{A}, \mathcal{B} \in \mathrm{LD}(\mathcal{K})$, we have:

$$(\mathcal{A} \cong \mathcal{B}) \;\Leftrightarrow\; (\Gamma(\mathcal{A}) \, E_0 \, \Gamma(\mathcal{B})).$$

In other words, (modulo technical details) we have a continuous reduction from $\cong \restriction LD(\mathcal{K})$ to the relation E_0.

Definition

Let E be an equivalence relation on the Cantor space. Let $\mathcal{K} = \{\mathcal{A}_i : i \in \omega\}$ be a family of countable L-structures.

We say that the family $\mathcal{K}$ is $\underline{E\text{-learnable}}$ if there is a continuous function $\Gamma\colon 2^\omega \to 2^\omega$ such that for all $\mathcal{A}, \mathcal{B} \in \mathrm{LD}(\mathcal{K})$, we have:

$$(\mathcal{A} \cong \mathcal{B}) \ \Leftrightarrow \ (\Gamma(\mathcal{A}) \ E \ \Gamma(\mathcal{B})).$$

Remark

In general, one could also consider E-learnability for uncountable families, but we will not discuss it here.

The first example: Increase of the learning power

We can identify the Cantor space with the space of all countable graphs: for $\alpha \in 2^\omega$, we have

$$G_\alpha \models \mathrm{Edge}(i,j) \ \Leftrightarrow \ \alpha(\langle i,j \rangle) = 1.$$

Then for a countable ordinal $\lambda > 0$, we define the following equivalence relation on 2^ω:

$$(\alpha \, R_\lambda \, \beta) \ \Leftrightarrow \ (\text{the graphs } G_\alpha \text{ and } G_\beta \text{ satisfy the same } \Sigma^{\mathrm{inf}}_\lambda \text{ sentences}).$$

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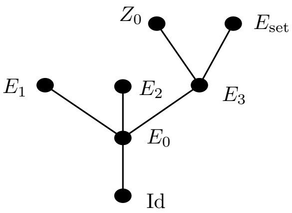
Proposition (essentially follows from B. 2017)

Let $\mathcal{K} = \{\mathcal{A}_i : i \in \omega\}$ be a family of countable L-structures. Then the following conditions are equivalent:

- The family $\mathcal{K}$ is R_λ-learnable.
- There are $\Sigma^{\mathrm{inf}}_\lambda$ sentences ψ_i, $i \in \omega$, such that

$$\mathcal{A}_i \models \psi_j \text{ if and only if } i = j.$$

Case study: Some benchmark Borel equivalence relations



(under continuous reductions)

- Id is the identity relation.
- By $\alpha^{[m]}$ we denote the m-th column of the real α.
  $(\alpha\, E_1\, \beta)$ if and only if
  $$(\forall^{\infty} m \in \omega)(\alpha^{[m]} = \beta^{[m]}).$$
- $(\alpha\, E_2\, \beta)$ if and only if
  $$\sum_{k=0}^{\infty} \frac{(\alpha \triangle \beta)(k)}{k+1} \; < \; \infty.$$
- $(\alpha\, E_3\, \beta)$ if and only if $(\forall m)(\alpha^{[m]}\ E_0\ \beta^{[m]})$.
- $(\alpha\, E_{\mathrm{set}}\, \beta)$ if and only if $\{\alpha^{[m]} : m \in \omega\} = \{\beta^{[m]} : m \in \omega\}$.
- $(\alpha\ Z_0\ \beta)$ if and only if $\alpha \triangle \beta$ has asymptotic density zero.

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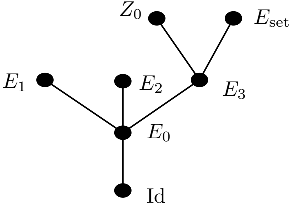

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Remark

If E is continuously reducible to F, then every E-learnable family is also F-learnable.

A syntactic characterization of E_3-learnability

$(\alpha \, E_3 \, \beta)$ if and only if $(\forall m)(\alpha^{[m]} \; E_0 \; \beta^{[m]})$.

Theorem 2 (BCS)

Let $\mathcal{K} = \{\mathcal{A}_i : i \in \omega\}$ be a family of countable L-structures. Then the following conditions are equivalent:

- The family $\mathcal{K}$ is E_3-learnable.
- There exists a family of $\Sigma_2^{\inf}$ sentences Θ such that:
  - (a) For every $\theta \in \Theta$, there exists a formula $\xi \in \Theta$ such that for every $\mathcal{A} \in \mathcal{K}$, we have $\mathcal{A} \models (\theta \leftrightarrow \neg\xi)$.

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  In other words, each pair $(\mathcal{A}_i, \mathcal{A}_j)$, for $i \neq j$, is "separated" by a property which is "Δ_2^{inf}-definable" inside $\mathcal{K}$.

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Every finite E_3-learnable family is already E_0-learnable.
There exists an infinite E_3-learnable family which is not E_0-learnable.

Learning-related reducibilities

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It is natural to consider the following learning-related reducibilities.

Definition

Let E and F be equivalence relations on 2^ω.

(1) $E \leq_{\mathrm{Learn}} F$ if every countable E-learnable family is also F-learnable.

(2) $E \leq^{<\omega}_{\mathrm{Learn}} F$ if every finite E-learnable family is also F-learnable.

It is clear that:

$$\text{continuous reducibility} \;\Rightarrow\; \leq_{\mathrm{Learn}} \;\Rightarrow\; \leq^{<\omega}_{\mathrm{Learn}}.$$

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Corollary 1, reformulated

We have $E_0 \equiv_{\mathrm{Learn}}^{<\omega} E_3$ and $E_0 <_{\mathrm{Learn}} E_3$. Hence, $\leq_{\mathrm{Learn}} \;\not\Leftarrow\; \leq_{\mathrm{Learn}}^{<\omega}$.

Finite families and benchmark relations

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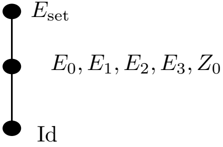
Theorem 3 (BCS)

With respect to $\leq^{<\omega}_{\mathrm{Learn}}$, we have:



In addition, the family $\{\omega, \zeta\}$ is E_{set}-learnable but not E_0-learnable.

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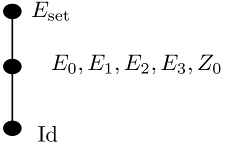

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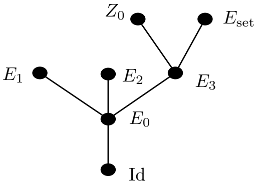

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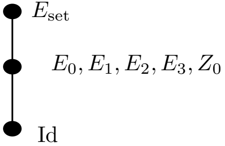

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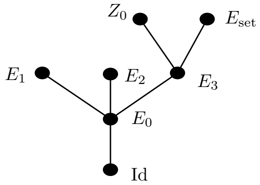

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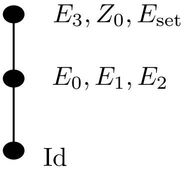


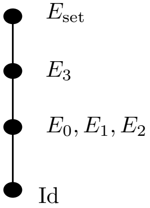
(under the reducibility $\leq^{<\omega}_0$ [R. Miller 2020])

Infinite families and benchmark equivalence relations

$E \leq_{\mathrm{Learn}} F$ if every countable E-learnable family is also F-learnable.

Theorem 4 (BCS)

With respect to $\leq_{\mathrm{Learn}}$, we have:



In particular, this shows that: $\leq_{\mathrm{Learn}} \;\not\Rightarrow$ Borel reducibility.

It is still open what happens with the $\leq_{\mathrm{Learn}}$-degree of Z_0.

Infinite families and benchmark equivalence relations

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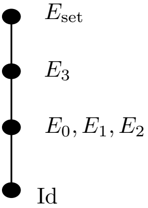
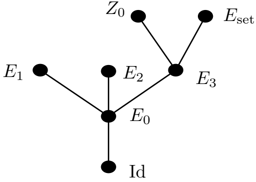
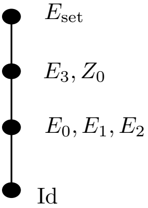


(under $\leq_{\mathrm{Learn}}$)



(under continuous reductions)



(under the reducibility $\leq_0^\omega$ [R. Miller 2020])

Further problems

Problem 1

What is the $\leq_{\mathrm{Learn}}$-degree of the benchmark relation Z_0? What about other popular benchmark relations?

Problem 2

Can one obtain a "nice" syntactic characterization of E_{set}-learnability?

Further problems

Problem 3

Obtain descriptive set-theoretic characterizations for *other* learning paradigms.

Example. A countable family $\mathcal{K} = \{\mathcal{A}_i : i \in \omega\}$ is $\mathbf{InfFin}$-learnable if it is $\mathbf{InfEx}$-learnable by a learner M which satisfies the following additional property: if $\mathcal{S} \cong \mathcal{A}_i$, then there is $k^* \in \omega$ such that

$$M(\mathcal{S} \upharpoonright l) = \begin{cases} ?, & \text{if } l < k^*, \\ i, & \text{if } l \geq k^*. \end{cases}$$

In other words, M never says wrong conjectures on the input $\mathcal{S}$.

Further problems

Problem 3

Obtain descriptive set-theoretic characterizations for *other* learning paradigms.

Example. A countable family $\mathcal{K} = \{\mathcal{A}_i : i \in \omega\}$ is $\mathbf{InfFin}$-learnable if it is $\mathbf{InfEx}$-learnable by a learner M which satisfies the following additional property: if $\mathcal{S} \cong \mathcal{A}_i$, then there is $k^* \in \omega$ such that

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In other words, M never says wrong conjectures on the input $\mathcal{S}$.

Proposition (BCS)

A family $\mathcal{K}$ is $\mathbf{InfFin}$-learnable if and only if there is a continuous function $\Gamma\colon 2^\omega \to 2^\omega$ such that:

- for any $\mathcal{A}, \mathcal{B} \in \mathrm{LD}(\mathcal{K})$, we have $\mathcal{A} \cong \mathcal{B}$ iff $\Gamma(\mathcal{A}) = \Gamma(\mathcal{B})$;
- the set $\Gamma(LD(\mathcal{K}))$ has no limit points.

In other words, $\mathcal{K}$ is Id-learnable with an additional topological property.

References

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