Countable Reductions, Borel Equivalence Relations, and Computable Structure Theory

Progress report on joint work with Meng-Che "Turbo" Ho & Julia Knight

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Borel reducibility and computable reducibility

Fundamental Definition

A *reduction* of an equivalence relation E on a set S to another equivalence relation F on a set T is a function $f : S \to T$ such that

$$(\forall (a, b) \in S \times S)\ [\ a\ E\ b \iff f(a)\ F\ f(b)\ ].$$

First example: E on 2^ω is *Borel reducible* to F on 2^ω (or other Polish spaces) if there exists a *Borel reduction* $f : 2^\omega \to 2^\omega$ of E to F.

Second example: E on ω is *computably reducible* to F on ω (or other subsets of ω) if there is a *computable reduction* $f : \omega \to \omega$ of E to F.

Related work by many researchers

Computable reducibility has arisen independently several times.

- Yuri L. Ershov (several works from the 1960's).
- Claudio Bernardi and Andrea Sorbi, "Classifying positive equivalence relations," JSL 48(3):529–538, 1983.
- Su Gao and Peter Gerdes, "Computably enumerable equivalence relations," Studia Logica, 67(1):27–59, 2001.
- Julia F. Knight, Sara Miller, and Michael Vanden Boom, "Turing computable embeddings," *JSL* 72(3):901–918, 2007.
- S. Buss, Y. Chen, J. Flum, S.-D. Friedman and M. Müller, "Strong isomorphism reductions . . .," *JSL* 76(4):1381-1402, 2011.
- Ekaterina B. Fokina and Sy-David Friedman, "On Sigma-1-1 equivalence relations over the natural numbers," MLQ, 2011.
- Sam Coskey, Joel D. Hamkins, & RGM, "The hierarchy of equivalence relations. . .," *Computability* 1:15-38, 2012.
- Lempp, J. Miller, Ng, San Mauro, and Sorbi, "Universal computably enumerable equivalence relations," *JSL* 79(1):60-88, 2014.

Notation

We usually assume that one can determine from the equivalence relations E and F whether these are relations on ω or on 2^ω (or other spaces). Our notation allows for various generalizations.

Notation

For relations on ω, $E \leq_{\boldsymbol{d}} F$ means that E is $\boldsymbol{d}$-computably reducible to F. Thus $E \leq_{\boldsymbol{0}} F$ denotes that a computable reduction of E to F exists.

For relations on 2^ω, $E \leq_B F$ means that E is Borel-reducible to F, i.e., that a Borel reduction of E to F exists. More specific versions, such as $E \leq_0 F$ or $E \leq_\alpha F$ will correspond to specific Borel functions, with the ordinal α denoting the number of jumps required to compute the reduction.

Superscripts (e.g., $\leq_{\boldsymbol{0}}^n$) will be added shortly for *bounded* reductions.

Same-minimum & same-maximum relations

Same minimum:

$$i \, E^{ce}_{\min} \, j \quad \iff \quad \min(W_i) = \min(W_j) \;\; (\text{or } W_i = W_j = \emptyset).$$

Same maximum:

$$i \, E^{ce}_{\max} \, j \quad \iff \quad \max(W_i) = \max(W_j) \in \omega \cup \{\pm\infty\}.$$

Coskey, Hamkins, and RGM showed that neither of these computably reduces to the other. For $E^{ce}_{max} \not\leq_{\mathbf{0}} E^{ce}_{min}$, complexity suffices.

Direct proof that $E^{ce}_{max} \not\leq_{\mathbf{0}} E^{ce}_{min}$ via f: Enumerate two sets W_i and W_j. Wait until $f(i)\downarrow$ and $f(j)\downarrow$, then make $\max(W_{i,s}) = \max(W_{j,s})$ iff $\min(W_{f(i),s}) \neq \min(W_{f(j),s})$.

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<u>Us</u> (E^{ce}_{max}) | <u>Them</u> (E^{ce}_{min})

W_i: | $W_{f(i)}$:

W_j: | $W_{f(j)}$:

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W_i:	0	$W_{f(i)}$:	
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W_i:	0, 1	$W_{f(i)}$:	n
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W_i:	0, 1	$W_{f(i)}$:	$n-1,\ n$
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W_i:	0, 1	$W_{f(i)}$:	$n-1,\ n$
W_j:	0, 1	$W_{f(j)}$:	n

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W_i:	0, 1, 2	$W_{f(i)}$:	$n-1,\; n$
W_j:	0, 1	$W_{f(j)}$:	$n-1,\; n$

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W_i:	$0, 1, 2, \ldots, n, n+1$	$W_{f(i)}$:	$0, \ldots, n-1, n$
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$E^{ce}_{min} \not\leq_{\mathbf{0}} E^{ce}_{max}$ is harder!

$E^{ce}_{min} \not\leq_{\mathbf{0}} E^{ce}_{max}$ (by CHM 2012), but we cannot prove this the same way.
Indeed, there exist computable total g, h such that

$$\forall i, j \ \left[\min(W_i) = \min(W_j) \quad \iff \quad \max(W_{g(i,j)}) = \max(W_{h(i,j)})\right].$$

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One can do the same for three arbitrary indices i, j, k, or for however many one likes. Thus E_{min}^{ce} is *finitarily reducible* to E_{max}^{ce}.

Definition

An *n-ary computable reduction* of E to F (making $E \leq_{\mathbf{0}}^{n} F$) consists of n computable functions $g_1, \ldots, g_n$ such that for every $(x_1, \ldots, x_n)$,

$$(\forall j < k \leq n)\ [x_j \ E \ x_k \iff g_j(\vec{x}) \ F \ g_k(\vec{x})].$$

E is *finitarily reducible* to F if this can be done uniformly for all $n \in \omega$.

$E_{min}^{ce} \not\leq_{\mathbf{0}} E_{max}^{ce}$ is harder!

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E is *finitarily reducible* to F if this can be done uniformly for all $n \in \omega$.

2-ary reducibility $\leq_{\mathbf{0}}^{2}$ is just the familiar many-one reducibility $E \leq_m F$.

Within *n*-ary reducibility...

A strange fact about E^{ce}_{max}.

Theorem (RGM-Ng)

E^{ce}_{max} is complete among all Π^0_2 equivalence relations under 3-ary reducibility $\leq^3_{\mathbf{0}}$, but not under 4-ary reducibility $\leq^4_{\mathbf{0}}$.

The proof uses the $=^{ce}$ relation on ω: $\quad i =^{ce} j \iff W_i = W_j$.

Since $=^{ce}$ is Π^0_2-complete under finitary reducibility $\leq^{<\omega}_{\mathbf{0}}$ (to be seen below), we show that $=^{ce} \leq^3_{\mathbf{0}} E^{ce}_{max}$, but that $=^{ce} \not\leq^4_{\mathbf{0}} E^{ce}_{max}$.

3-ary reduction: $=^{ce} \leq^3_{\mathbf{0}} E^{ce}_{max}$

Define $g(i,j,k) = (\hat{i},\hat{j},\hat{k})$: find the least difference between W_i and W_j, between W_i and W_k, and between W_j and W_k at each stage. Each time one of these differences changes from the previous stage, add more elements to $W_{\widehat{i}}$, $W_{\widehat{j}}$, and $W_{\widehat{k}}$ to reflect the new situation.

- If $W_i = W_j = W_k$, then all three sets we build wind up with max $+\infty$.
- If $W_i = W_j \neq W_k$, then the least difference between W_i and W_j keeps changing, so $W_{\widehat{i}}$ and $W_{\widehat{j}}$ both have max $+\infty$. But eventually the least difference between W_k and each of them stabilized, so $W_{\widehat{k}}$ has finite maximum.
- If the three sets are all distinct, then by some stage their (pairwise) least differences had all appeared, and from that stage on, the maxima of $W_{\widehat{i}}$, $W_{\widehat{j}}$, and $W_{\widehat{k}}$ will be the same three distinct (finite) values.

No 4-ary reduction: $=^{ce} \not\leq^4_{\mathbf{0}} E^{ce}_{max}$

Given a potential 4-ary reduction g, we build four c.e. sets. W_i and W_j use only odd numbers, while W_m and W_n use only even numbers. By making W_i and W_j equal whenever their images have distinct maxima at some stage, and unequal whenever their images have the same maxima, we drive those maxima to $+\infty$, leaving $W_i = W_j$. With W_m and W_n, we do exactly the same. So all four images have maximum $+\infty$, yet $W_i \neq W_m$.

This reflects the fact that in E^{ce}_{max}, there is exactly one equivalence class (namely **Inf**) which is Π^0_2-complete as a set, whereas relations such as $=^{ce}$ have infinitely many Π^0_2-complete classes. The putative 4-ary reduction had to use that single class **Inf** to compare W_i with W_j, and also to compare W_m with W_n.

Π^0_n-Completeness

Ianovski, RGM, Ng, and Nies have shown that for $n \geq 2$, no Π^0_n equivalence relation on ω can be complete among Π^0_n-ER's under computable reducibility. However....

Theorem (RGM-Ng)

For every oracle set $X \subseteq \omega$, the equivalence relation $E^X_=$ given by

$$i \; E^X_= \; j \quad \iff \quad W^X_i = W^X_j$$

is complete among all Π^X_2 equivalence relations under finitary (computable) reducibility $\leq^{<\omega}_{\mathbf{0}}$.

Corollary

For every $n \geq 0$, the Π^0_{n+2} equivalence relation $E^{\emptyset^{(n)}}_=$ (i.e. equality on Σ_{n+1} sets) is complete among Π^0_{n+2} ER's under finitary reducibility.

Finitary and d-computable reducibility

Same theorem, repeated (RGM-Ng)

For every oracle set $X \subseteq \omega$, the equivalence relation $E^X_=$ given by

$$i\ E^X_=\ j \quad \iff \quad W^X_i = W^X_j$$

is complete among all Π^X_2 equivalence relations under finitary (computable) reducibility $\leq^{<\omega}_{\mathbf{0}}$.

Corollary

For every Turing degree $\boldsymbol{d}$, there exist equivalence relations E and F such that $E\ \leq^{<\omega}_{\mathbf{0}}\ F$, but $E\ \not\leq_{\boldsymbol{d}}\ F$. (That is, there is no $\boldsymbol{d}$-computable reduction from E to F, but there is a computable finitary reduction.)

Proof: Let F be $E^{\boldsymbol{d}}_=$, which is $\Pi^{\boldsymbol{d}}_2$-complete under finitary reducibility, but $\Pi^{\boldsymbol{d}}_2$-incomplete under $\boldsymbol{d}$-computable reducibility, by a relativization of our previous result.

Countable computable reducibility

One could also ask for a computable function f that, for every countable list $x_0, x_1, x_2, \ldots$ of elements of ω, lists out $y_0, y_1, y_2, \ldots$ such that $x_i \; E \; x_j$ iff $y_i \; F \; y_j$. (This f would really be a Turing functional Φ, which is given the function $i \mapsto x_i$ as an oracle and computes the function $i \mapsto y_i$.) In this case we might write $E \leq_{\mathbf{0}}^{\omega} F$.

But if this exists, then by giving Φ the computable oracle $i \mapsto i$, we would get a full computable reduction. Thus $E \leq_{\mathbf{0}}^{\omega} F$ iff $E \leq_{\mathbf{0}} F$.

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But maybe if we go back to ER's on 2^{ω} ......

Extending the paradigm to 2^ω

Now let E, F be ER's on 2^ω. What happens in Borel reducibility?

Definition

E is ω-reducible to F (written $E \leq_0^\omega F$) if there exists a Turing functional Φ such that whenever $A_0, A_1, A_2, \ldots \in 2^\omega$ and $A = \oplus_i A_i$,

$$(\forall j, k \in \omega)\ \left[A_j\ E\ A_k \iff \left(\Phi^A\right)^{(j)}\ F\ \left(\Phi^A\right)^{(k)}\right].$$

We also have weaker versions, analogous to the earlier ones:
E is n-arily reducible to F (written $E \leq_0^n F$) if there exists a Turing functional Φ such that whenever $A_1, \ldots, A_n \in 2^\omega$

$$(\forall j < k \leq n)\ \left[A_j\ E\ A_k \iff \left(\Phi^{A_1 \oplus \cdots \oplus A_n}\right)^{(j)}\ F\ \left(\Phi^{A_1 \oplus \cdots \oplus A_n}\right)^{(k)}\right].$$

E is *finitarily reducible* to F, $E \leq_0^{<\omega} F$, if this can be done uniformly $\forall n$.

Standard Borel equivalence relations, under $\leq_B$

$$A\ E_0\ B \iff |A \triangle B| < \infty \text{ (finite-difference relation, a.k.a. } =^*\text{)}$$

$$A\ E_1\ B \iff |\{n : (A)^n \neq (B)^n\}| < \infty \text{ (differ on } < \infty \text{ columns)}$$

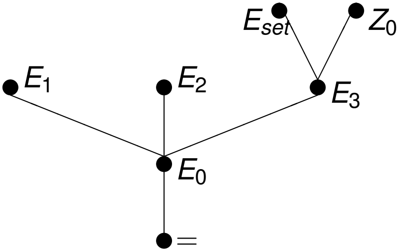
$$A\ E_2\ B \iff \sum_{n \in A \triangle B} \frac{1}{n+1} < \infty.$$

$$A\ E_3\ B \iff (\forall n)\ |(A)^n \triangle (B)^n| < \infty.$$

$$A\ E_{set}\ B \iff \{(A)^n : n \in \omega\} = \{(B)^n : n \in \omega\} \text{ (same columns).}$$

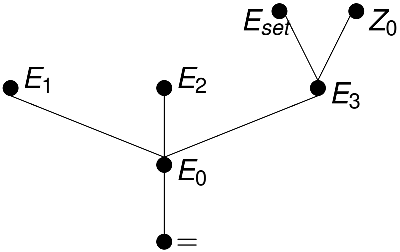
$$A\ Z_0\ B \iff (A \triangle B) \text{ has asymptotic density 0.}$$

$\leq_B$:



Comparing $\leq_B$, $\leq_0^\omega$, and $\leq_0^{<\omega}$

$\leq_B$



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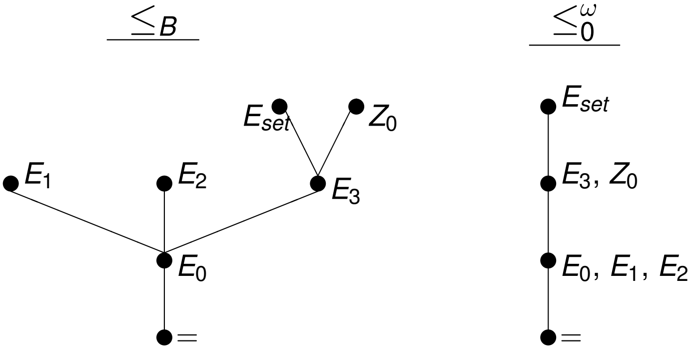
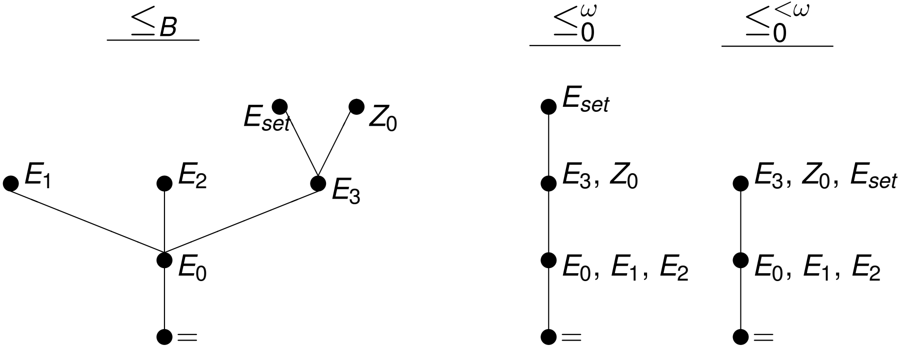


Comparing $\leq_B$, $\leq_0^\omega$, and $\leq_0^{<\omega}$



Complexity: $=$ is Π^0_1;
E_0, E_1, and E_2 are all Σ^0_2;
E_3, Z_0, and E_{set} are all Π^0_3.

Details: R. Miller, Computable reducibility for Cantor space, chapter in *Structure and Randomness in Computability and Set Theory*, eds. D. Cenzer, C. Porter, & J. Zapletal (World Scientific, 2020), 155–196.

Computable structure theory (joint with Ho-Knight)

All structures here will have domain ω. Using a fixed Gödel coding, we identify such a structure A with its atomic diagram $D(A)$, viewed as an element of 2^ω.

Important classes of structures:

$$\mathfrak{TFab}_r = \{D(A) : A \text{ is a torsion-free Abelian group of rank } r\}$$

$$\mathfrak{TD}_r = \{D(F) : F \text{ is a field of tr. degree } r \text{ over } \mathbb{Q}\}.$$

Equivalently, A is a full-rank additive subgroup of $(\mathbb{Q}^r, +, \vec{0})$, and F is a full-degree subfield of the algebraic closure $\overline{\mathbb{Q}(t_1, \ldots, t_r)}$. However, each A and F is presented as a *structure*. Divisibility of elements of A is $\Sigma_1^{D(A)}$, and existence of roots of polynomials in F is $\Sigma_1^{D(F)}$.

We view $\mathfrak{TFab}_r$ and $\mathfrak{TD}_r$ as topological subspaces of 2^ω, equating A with $D(A)$, and place the ER of isomorphism on each subspace.

Theorem of Hjorth & Thomas

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For every $r > 0$, isomorphism on $\mathfrak{TFab}_r$ lies strictly below isomorphism on $\mathfrak{TFab}_{r+1}$ under Borel (!) reducibility.

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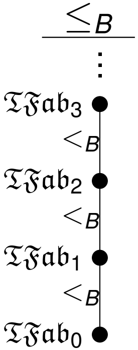
In the language of Knight et al, this implies that there is no *Turing-computable embedding* of $\mathfrak{TFab}_{r+1}$ into $\mathfrak{TFab}_r$. In contrast, the reverse embedding is easily built:

$$(\forall G, H \in \mathfrak{TFab}_r)\ [G \cong H \iff (G \oplus \mathbb{Z}) \cong (H \oplus \mathbb{Z})].$$

Computing a presentation of $(G \oplus \mathbb{Z})$ uniformly from each $G \in \mathfrak{TFab}_r$ is easy, and the equivalence exactly satisfies the definition of a TC-embedding Φ of a class $\mathfrak{K}$ into another $\mathfrak{K}'$:

$$(\forall \mathcal{A}, \mathcal{B} \in \mathfrak{K})\ \left[\Phi^{\mathcal{A}}, \Phi^{\mathcal{B}} \in \mathfrak{K}',\ \text{with } \mathcal{A} \cong \mathcal{B} \iff \Phi^{\mathcal{A}} \cong \Phi^{\mathcal{B}}\right].$$

Comparing $\leq_B$ and $\leq_0^\omega$ on $\mathfrak{TFab}$



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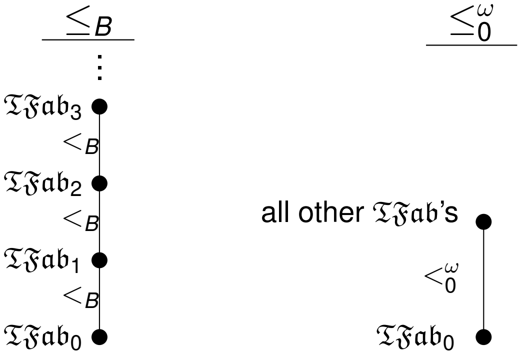

Fact: Every $\mathfrak{TFab}_r$ with $r > 0$ has a countable computable reduction to $\mathfrak{TFab}_1$.

This is surprising. Isomorphism on $\mathfrak{TFab}_1$ seems simpler than the others! (Cf. Baer, Kurosh, Mal'cev, finally Hjorth-Thomas.)

The isomorphism relation on $\mathfrak{TFab}_r$

For $G \in \mathfrak{TFab}_1$, fix any nonzero $x \in G$ and define

$$I_G = \{(p, n) \in \mathbb{P} \times \mathbb{N} : (\exists y \in G)\ p^n y = x\}.$$

Now $G \cong H$ iff $I_G\ E_0\ I_H$. Here isomorphism is Σ^0_3 (and Σ^0_3-complete).

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For $r > 1$ and $G, H \in \mathfrak{TFab}_r$, $G \cong H$ iff

$$(\exists \text{ bases } \vec{b} \in G^r, \vec{c} \in H^r)(\forall \vec{q} \in \mathbb{Q}^r)\left[\sum q_i b_i \in G \iff \sum q_i c_i \in H\right].$$

This seems more complicated, but it is again Σ_3. Indeed, the isomorphism problem on computable $\mathfrak{TFab}_r$ groups is Σ_3-complete.

So it is plausible that the countable computable reduction works.

Fields of finite transcendence degree

$\mathfrak{TD}_r$ is very similar, except for $r = 0$. Isomorphism is again Σ^0_3; for a fixed transcendence basis $\{x_1, \ldots, x_r\}$ of F, the isomorphism type is determined by $\{f \in \mathbb{Z}[X_1, \ldots, X_r, Y] : f(\vec{x}, Y)$ has a root in $F\}$. Two fields are isomorphic iff they have bases over which these index sets are equal.

When $r = 0$: $\mathfrak{TD}_0$ is nontrivial (as opposed to $\mathfrak{TFab}_0$), but now isomorphism is Π^0_2, as the only possible basis is $\emptyset$.

Going upwards in fields

Conjecture (HKM)

$\mathfrak{TD}_r \leq_0 \mathfrak{TD}_{r+1}$.

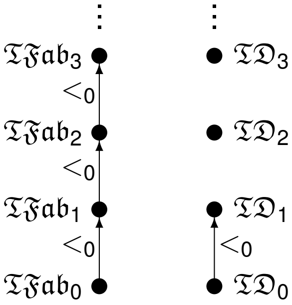
For $r = 0$, this is immediate. If $E, F \in \mathfrak{TD}_0$, then all their elements are algebraic over $\mathbb{Q}$, and no other elements of the purely transcendental extension $E(t)$ are algebraic over $\mathbb{Q}$. Thus $E \cong F \iff E(t) \cong F(t)$.

For $r > 0$, we conjecture that the same procedure succeeds.

A separate attempt would build $E(t)$ from E and then adjoin the entire algebraic closure of $\mathbb{Q}(t)$. This succeeds in the situation $E \cap \overline{\mathbb{Q}} = F \cap \overline{\mathbb{Q}}$. However, if E and F have nonisomorphic algebraic parts, this attempt obliterates that distinction. One can salvage the fact that $\mathfrak{TD}_1 \leq_0 \mathfrak{TD}_2 \times \mathfrak{TD}_0$.

Going rightwards

Current picture for full computable reducibility $\leq_0$:



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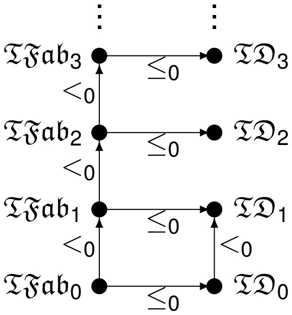


Theorem (HKM)

For every r, there is a full computable reduction $\mathfrak{TFab}_r \leq_0 \mathfrak{TD}_r$.

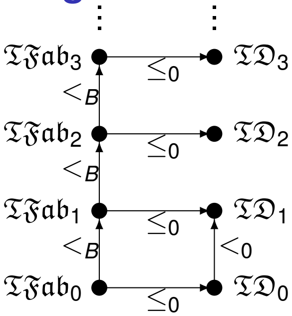
$\mathfrak{TFab}_r \leq_0 \mathfrak{TD}_r$

The idea of this reduction is simple: given $G \in \mathfrak{TFab}_r$, we write each $v \in G$ as $(a_1, \ldots, a_r)$, with respect to the first basis we find for G. We then build a field $F = \Phi^G \in \mathfrak{TD}_r$ with basis $\{X_1, \ldots, X_r\}$, and include an element $X_1^{a_1} \cdots X_r^{a_r}$ to represent v. Details:

- If we guess wrong about a basis $\vec{u}$ for G – maybe it turns out that $u_3 = \frac{2}{5}u_1 - \frac{7}{3}u_2$ – then the old X_3 becomes $X_1^{\frac{2}{5}} X_2^{-\frac{7}{3}}$ and we add a new X_3 to F to represent the next element that appears independent over $\{u_1, u_2\}$.
- All powers of each X_i are positive in a fixed real closed field.
- To build F, close under the field operations.
- If we had used a different basis $\vec{w}$ of G, there would be an $A \in GL_r(\mathbb{Q})$ expressing $\vec{u}$ w.r.t. $\vec{w}$. Applying A to the exponents of the monomials gives a field isomorphism between the outputs.

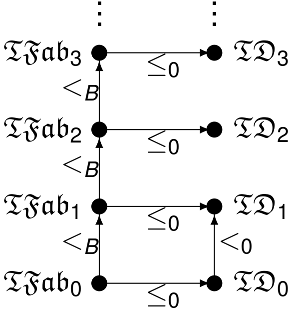
The hard part is to show that $\Phi^G \cong \Phi^H \implies G \cong H$.

Anything more?



Hjorth and Thomas showed that there is no Borel reduction downward in the left column. Consequently, for every $r > 0$, either $\mathfrak{TD}_{r+1} \not\leq_B \mathfrak{TD}_r$ or $\mathfrak{TD}_r \not\leq_B \mathfrak{TFab}_r$: there is cannot be a downward reduction in the right column and also a leftward reduction. We conjecture that both fail.

Anything more?



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Theorem (HKM)

For every r, there is a countable computable reduction $\mathfrak{TD}_r \leq_0^\omega \mathfrak{TFab}_1$.

$\mathfrak{TD}_r \leq_0^\omega \mathfrak{TFab}_1$

Given fields $F_0, F_1, \ldots$ from $\mathfrak{TD}_r$, we guess at the first basis $\vec{x}_i$ in each F_i, "resetting" the construction of the corresponding $G_i \subseteq \mathbb{Q}$ whenever we reset the guess at a basis of F_i.

To each $i < j$, we assign primes p_{ijk} for all k, and use chips to approximate whether any of the first k tuples in F_j can be the image of $\vec{x}_i$ under an isomorphism $F_i \to F_j$.

- If k gets infinitely many chips, then 1 is infinitely divisible by p_{ijk} in every group G_m.
- Otherwise, 1 is divisible in G_i by one more power of p_{ijk} than in G_j. We also consider each of the fields F_m with $m < i + j + k$: guessing at the isomorphism relations among all these fields (including G_i and G_j), and giving the 1 in G_m the appropriate amount of divisibility by p_{ijk}.

This works!

If $F_i \cong F_j$, then there are only finitely many k where divisibility by p_{ijk} comes out different, and there are only finitely many (i', j', k') where it is left out of the guessing entirely. On each of these the divisibility of 1 differs only by finitely many powers of p_{ijk}, and for all other prime powers p^n, $\frac{1}{p^n}$ lies in G_i iff it lies in G_j. Hence $G_i \cong G_j$.

If $F_i \ncong F_j$, then $\forall k$ the 1 in G_i is divisible by one less power of p_{ijk} than the 1 in G_j, leaving $G_i \ncong G_j$.

A further point

The reduction $\mathfrak{TFab}_r \leq_0 \mathfrak{TD}_r$, mapping $\vec{a} \in G$ to $X_1^{a_1} \cdots X_r^{a_r} \in F$, is not just a computable reduction. Indeed, it extends to a computable functor, where the categories in question are $\mathfrak{TFab}_r$ under isomorphisms and $\mathfrak{TD}_r$ under isomorphisms. There is a Turing functional Ψ that, given any isomorphism $g : G_0 \to G_1$ in $\mathfrak{TFab}_r$, outputs an isomorphism $f : F_0 \to F_1$ for the corresponding fields, in a functorial way (respecting composition and the identity isomorphism).

For other Borel reductions on isomorphism relations in computable structure theory, one can investigate the same questions. Does the reduction extend to a functor? And if so, how many jumps (of the atomic diagrams of the structures) are needed in order to compute the entire functor?

Does this generalize?

Thinking of the notion of computable functor, we ask whether anything similar applies to the known ER's (or others) in the Borel theory. These ER's do not ask Σ^1_1-hard questions, but isomorphism in $\mathfrak{TFab}$ and $\mathfrak{TD}$ is not Σ^1_1-hard either. For example, for the relation $A\ Z_0\ B$ defined by $(A \triangle B)$ has asymptotic density 0, one could say that a function $f : \omega \to \omega$ *witnesses* that $A\ Z_0\ B$ if f bounds the convergence to 0:

$$(\forall d)(\forall n > f(d))\ \frac{|(A \triangle B) \cap \{0, \ldots, n-1\}|}{n} \leq \frac{1}{d}.$$

Can one give a computable reduction Φ of E_3 to Z_0 for which, given any $A\ E_3\ B$ and the function $f(n) = \max((A)^n \triangle (B)^n)$, some uniform procedure computes a witness to $\Phi^A\ Z_0\ \Phi^B$?

Conclusions and questions

The proof that $\mathfrak{TD}_r \leq_0^\omega \mathfrak{TFab}_1$ is totally in line with traditional computable structure theory. We believe that when $\leq_0$ and $\leq_0^\omega$ differ, methods beyond those traditional approaches will be required, as in the Hjorth-Thomas theorem. Somehow here the uncountability of the space becomes crucial.

What happens if, instead of considering only computable reductions, we allow reductions that use $(D(A))'$ to compute the output for A? Or $(D(A))^{(\alpha)}$?

For isomorphism or other Σ_1^1 relations, what sort of functoriality is possible, and how effective can it be?

How do bounded reducibilities such as $\leq_0^\omega$ play into all of this? Computable reductions are continuous. Is there any topological notion analogous to countable computable reducibility?