Aspects of Hausdorff Dimension

Theodore A. Slaman
(joint with Jan Reimann)

University of California Berkeley

Abstract

We will describe how the perspectives of Recursion Theory and Set Theory suggest lines of investigation into Geometric Measure Theory. We will discuss the existence problem for sets of strong gauge dimension zero, which is a property generalizing that of strong measure zero.

Gauge Functions and General Hausdorff Dimension

Definition

A *gauge function* is a function $h : (0, \infty) \to (0, \infty)$ which has the following properties:

- continuous
- increasing
- $\lim_{t \to 0^+} h(t) = 0$

Example

$h(t) = t^s$, for $s > 0$.

Gauge Functions and General Hausdorff Dimension

Definition

Let h be a gauge function. For a set $E \subseteq 2^\omega$ (or ω^ω, $\mathbb{R}^d$ etc.), define

$$H^h(E) = \lim_{\delta \to 0} \inf_{\substack{E \subseteq \cup F_i \\ \max d(F_i) < \delta}} \sum_{i=1}^{\infty} h(d(F_i))$$

where $\{F_i\}$ is a sequence of closed (open) sets covering E and $d(F_i)$ is the diameter of F_i.

Gauge Functions and General Hausdorff Dimension

Definition

Let h be a gauge function. For a set $E \subseteq 2^\omega$ (or ω^ω, $\mathbb{R}^d$ etc.), define

$$H^h(E) = \lim_{\delta \to 0} \inf_{\substack{E \subseteq \cup F_i \\ \max d(F_i) < \delta}} \sum_{i=1}^{\infty} h(d(F_i))$$

where $\{F_i\}$ is a sequence of closed (open) sets covering E and $d(F_i)$ is the diameter of F_i.

- When $h(t)$ is t^s, $H^h = H^s$ is the usual s-dimensional Hausdorff outer measure of E.
- Gauge functions provide a more finely graded calibration of measure and thereby of dimension than is given by the family $\{t \mapsto t^s : s \in [0, 1]\}$.

Gauge Functions and General Hausdorff Dimension

Definition

Write $h \prec g$ to indicate that $\lim_{t \to 0^+} \frac{g(t)}{h(t)} = 0$.

Note, $h \prec g$ means that is it easier for a set to be H^g-null than it is to be H^h-null.

Example

$t^{\log(2)/\log(3)} \prec t^1$, since $\lim_{t \to 0^+} \frac{t}{t^{\log(2)/\log(3)}} = 0$.

Gauge Functions and General Hausdorff Dimension

The Hausdorff dimension of a set $A \subset \mathbb{R}$ is the number d such that whenever $d_0 < d < d_1$, $H^{d_0}(A) = \infty$ and $H^{d_1} = 0$.

Example

The Cantor middle-third set, which has dimension $\log(2)/\log(3)$ is null with respect to linear (Lebesgue) measure.

Gauge Functions and General Hausdorff Dimension

The clarity of Hausdorff dimension transfers only partially to gauge measures.

Gauge Functions and General Hausdorff Dimension

The clarity of Hausdorff dimension transfers only partially to gauge measures.

Similarities:

- If $H^h(A)$ is finite and $h \prec g$ then $H^g(A) = 0$.

Gauge Functions and General Hausdorff Dimension

The clarity of Hausdorff dimension transfers only partially to gauge measures.

Similarities:

- If $H^h(A)$ is finite and $h \prec g$ then $H^g(A) = 0$.
- If $H^h(A)$ is not zero and $j \prec h$ then $H^j(A)$ is infinite.

Gauge Functions and General Hausdorff Dimension

The clarity of Hausdorff dimension transfers only partially to gauge measures.

Similarities:

- If $H^h(A)$ is finite and $h \prec g$ then $H^g(A) = 0$.
- If $H^h(A)$ is not zero and $j \prec h$ then $H^j(A)$ is infinite.

Difference:

- (Besicovitch 1956) If $H^h(A) = 0$ then there is a j with $j \prec h$ such that $H^j(A) = 0$.

Sets of non-σ-finite measure

Definition

A set A is σ-finite for H^h iff A is a countable union of sets A_i, such that each $H^h(A_i)$ is finite.

Sets of non-σ-finite measure

Definition

A set A is σ-finite for H^h iff A is a countable union of sets A_i, such that each $H^h(A_i)$ is finite.

Improved observation from previous slide:

- If $H^h(A)$ is not zero and $j \prec h$ then A is non-σ-finite for $H^j(A)$.

Capacitability

Theorem (Davies 1956 for x^s, Sion and Sjerve 1962)

If E is analytic and is non-σ-finite for H^h, then there is a compact subset of E that is non-σ-finite for H^h.

Sets of Strong Dimension h

Definition

A set E has *strong dimension h* iff

$$\forall f[f \prec h \Rightarrow H^f(E) = \infty]$$

$$\forall g[h \prec g \Rightarrow H^g(E) = 0]$$

As a limiting case, E has strong dimension 0 iff for all g, $H^g(E) = 0$.

Sets of Strong Dimension h

Definition

A set E has *strong dimension h* iff

$$\forall f[f \prec h \Rightarrow H^f(E) = \infty]$$

$$\forall g[h \prec g \Rightarrow H^g(E) = 0]$$

As a limiting case, E has strong dimension 0 iff for all g, $H^g(E) = 0$.

Example

A line segment within the plane has strong dimension 1.

Sets of Strong Dimension h

Theorem (Besicovitch 1956, generalized Rogers 1962)

If E is compact and is non-σ-finite for H^h, then there is a g such that $h \prec g$ and E is non-σ-finite for H^g.

- Thus, if E is compact then E cannot have strong dimension h and be non-σ-finite for H^h.
- By the capacitability theorem, the same is true if E is analytic.

Sets of Strong Dimension h

Theorem (Besicovitch 1956, generalized Rogers 1962)

If E is compact and is non-σ-finite for H^h, then there is a g such that $h \prec g$ and E is non-σ-finite for H^g.

- Thus, if E is compact then E cannot have strong dimension h and be non-σ-finite for H^h.
- By the capacitability theorem, the same is true if E is analytic.

It would be interesting to find a proof of this theorem using effective methods: *Is there a point-to-set formulation of a set's not having σ-finite measure for H^h?*

Sets of Strong Dimension *h*

Theorem (Besicovitch 1963)

If CH then there is a set $E \subset \mathbb{R}^2$ such that E has strong linear dimension and is not σ-finite for linear measure.

Sets of Strong Dimension h

Theorem (Besicovitch 1963)

If CH then there is a set $E \subset \mathbb{R}^2$ such that E has strong linear dimension and is not σ-finite for linear measure.

Theorem (Combining Besicovitch 1963 with Erdős, Kunen and Mauldin 1981)

If $V = L$ there there is a Π^1_1 set $E \subseteq \mathbb{R}^2$ such that E has strong linear dimension and is not σ-finite for linear measure.

Borel Conjecture

Definition

A set $E \subseteq \mathbb{R}$ has *strong measure 0* iff for any sequence of positive real numbers $\{\epsilon_i\}$ there is a sequence of open intervals $\{O_i\}$ such that for each i, O_i has length ϵ_i, and $E \subseteq \cup_{i=1}^{\infty} O_i$.

Borel Conjecture

Definition

A set $E \subseteq \mathbb{R}$ has *strong measure 0* iff for any sequence of positive real numbers $\{\epsilon_i\}$ there is a sequence of open intervals $\{O_i\}$ such that for each i, O_i has length ϵ_i, and $E \subseteq \cup_{i=1}^{\infty} O_i$.

Borel (1919) conjectured that strong measure 0 implies countable (BC).

Borel Conjecture

Definition

A set $E \subseteq \mathbb{R}$ has *strong measure 0* iff for any sequence of positive real numbers $\{\epsilon_i\}$ there is a sequence of open intervals $\{O_i\}$ such that for each i, O_i has length ϵ_i, and $E \subseteq \cup_{i=1}^{\infty} O_i$.

Borel (1919) conjectured that strong measure 0 implies countable (BC).

Theorem

- *(Sierpiński 1928) CH implies that there is an uncountable set of strong measure 0.*
- *(Laver 1976) $Con(ZFC)$ implies $Con(ZFC + BC)$.*

Borel Conjecture

Theorem (Besicovitch 1955)

A set E has strong dimension 0 iff it has strong measure 0.

Borel Conjecture

Theorem (Besicovitch 1955)

A set E has strong dimension 0 iff it has strong measure 0.

Theorem (Another variation on Besicovitch 1963)

$\neg BC$ implies that there is a subset of $\mathbb{R}^2$ which has strong linear dimension and which is not σ-finite for linear measure.

A Challenge

Question

Does the Borel Conjecture imply that there do not exist f and E such that E has strong dimension f and E is not σ-finite for H^f?

A Challenge

Question

Does the Borel Conjecture imply that there do not exist f and E such that E has strong dimension f and E is not σ-finite for H^f?

The conceptual challenge is to overcome the intractability of the property that A is non-σ-finite for H^h.

Understanding Non-σ-finiteness

A case study

Consider Π^0_1 subsets of $2^\omega \times 2^\omega$ and linear measure H^1.

Understanding Non-σ-finiteness

A case study

Consider Π^0_1 subsets of $2^\omega \times 2^\omega$ and linear measure H^1.

Exercise

The set of indices for Π^0_1 subsets C of $2^\omega \times 2^\omega$ such that $H^1(C) \neq 0$ is arithmetic.

By the compactness of $2^\omega \times 2^\omega$, we can assume that all the open covers in the definition of $H^1(C)$ are finite, which means that the prima facie definition of "$H^1(C) \neq 0$" can be expressed arithmetically.

Understanding Non-σ-finiteness

Definition

Let $N\sigma F$ be the of indices for Π_1^0 subsets C of $2^\omega \times 2^\omega$ such that C is non-σ-finite for H^1

Understanding Non-σ-finiteness

Definition

Let $N\sigma F$ be the of indices for Π^0_1 subsets C of $2^\omega \times 2^\omega$ such that C is non-σ-finite for H^1

Theorem

$N\sigma F$ is Σ^1_1-complete.

Understanding Non-σ-finiteness

Definition

Let $N\sigma F$ be the of indices for Π^0_1 subsets C of $2^\omega \times 2^\omega$ such that C is non-σ-finite for H^1

Theorem

$N\sigma F$ is Σ^1_1-complete.

Here is an analogous but more familiar situation.

Exercise

The set of indices for Π^0_1 subsets C of 2^ω such that C is uncountable Σ^1_1-complete.

Use Cantor's theorem: C is uncountable iff C has a perfect subset.

Understanding Non-σ-finiteness

$N\sigma F$ is Σ^1_1

The ingredients in the proof of Davies's (1956) theorem about capacitability of non-σ-finiteness entail the following:

> C is non-σ-finite for H^1 iff there is perfect tree of closed sets such that each path corresponds to a closed set of H^1-positive measure.

It follows that $N\sigma F$ is a Σ^1_1 set.

Understanding Non-σ-finiteness

$N\sigma F$ is Σ^1_1-hard

Davies's insight above points the way toward proving Σ^1_1-hardness:

- Consider closed sets that are disjoint unions of sets of finite H^1-measure.
- The canonical subset of $2^\omega \times 2^\omega$ of finite positive H^1-measure is a line segment.

Understanding Non-σ-finiteness

$N\sigma F$ is Σ^1_1-hard

First note that whether a tree $T \subset \omega^{<\omega}$ has an infinite path is Σ^1_1-complete condition, so it is sufficient to reduce that property to $N\sigma F$:

First note that whether a tree $T \subset \omega^{<\omega}$ has an infinite path is Σ^1_1-complete condition, so it is sufficient to reduce that property to $N\sigma F$:

Given T, build a closed subset C of $2^\omega \times 2^\omega$ so that the following dichotomy holds.

- If T has an infinite path then C is a union of uncountably many disjoint line segments.
- If T does not have an infinite path, then C is a union of countably many line segments.

The End